Chapter 18

A First Look at Stochastic Integrals with the Wiener Process

Section 18.1 touches briefly on the martingale characterization of the Wiener process.

Section 18.2 gives a heuristic introduction to stochastic integrals, via Euler's method for approximating ordinary integrals.

18.1 Martingale Characterization of the Wiener Process

Because the Wiener process is a Lévy process (Example 139), it is self-similar in the sense of Definition 147. That is, for any $a > 0$, $W(at) \stackrel{d}{=} a^{1/2}W(t)$. In fact, if we define a new process W_a through $W_a(t,\omega) = a^{-1/2}W(at,\omega)$, then W_a is itself a Wiener process. Thus the whole process is self-similar. This is only one of several sorts of self-similarities in the Wiener process. Another is sometimes called *spatial homogeneity*: W_τ, defined through $W_\tau(t,\omega) = W(t+\tau,\omega) - W(\tau, omega)$ is also a Wiener process. That is, if we "re-zero" to the state of the Wiener process $W(\tau)$ at an arbitrary time τ, the new process looks just like the old process. $-W(t)$, obviously, is also a Wiener process.

Related to these properties is the fact that $W^2(t) - t$ is a martingale with respect to $\left\{\mathcal{F}_t^W\right\}$. (This is easily shown with a little algebra.) What is more surprising is that this is enough to characterize the Wiener process.

Theorem 224 (Martingale Characterization of the Wiener Process) *If $M(t)$ is a continuous martingale, and $M^2(t) - t$ is also a martingale, then $M(t)$ is a Wiener process.*

There are some very clean proofs of this theorem[1] — but they require us to use stochastic calculus! Doob (1953, pp. 384ff) gives a proof which does not, however. The details of his proof are messy, but the basic idea is to get the central limit theorem to apply, using the martingale property of $M^2(t) - t$ to get the variance to grow linearly with time and to get independent increments, and then seeing that between any two times t_1 and t_2, we can fit arbitrarily many little increments so we can use the CLT.

We will return to this result as an illustration of the stochastic calculus (Theorem 247).

18.2 A Heuristic Introduction to Stochastic Integrals

Euler's method is perhaps the most basic method for numerically approximating integrals. If we want to evaluate $I(x) \equiv \int_a^b x(t)dt$, then we pick n intervals of time, with boundaries $a = t_0 < t_1 < \ldots t_n = b$, and set

$$I_n(x) = \sum_{i=1}^{n} x\left(t_{i-1}\right)\left(t_i - t_{i-1}\right)$$

Then $I_n(x) \to I(x)$, if x is well-behaved and the length of the largest interval $\to 0$. If we want to evaluate $\int_{t=a}^{t=b} x(t)dw$, where w is another function of t, the natural thing to do is to get the derivative of w, w', replace the integrand by $x(t)w'(t)$, and perform the integral with respect to t. The approximating sums are then

$$\sum_{i=1}^{n} x\left(t_{i-1}\right) w'\left(t_{i-1}\right)\left(t_i - t_{i-1}\right) \tag{18.1}$$

Alternately, we could, if $w(t)$ is nice enough, approximate the integral by

$$\sum_{i=1}^{n} x\left(t_{i-1}\right)\left(w\left(t_i\right) - w\left(t_{i-1}\right)\right) \tag{18.2}$$

even if w' doesn't exist.

(You may be more familiar with using Euler's method to solve ODEs, $dx/dt = f(x)$. Then one generally picks a Δt, and iterates

$$x(t + \Delta t) = x(t) + f(x)\Delta t \tag{18.3}$$

from the initial condition $x(t_0) = x_0$, and uses linear interpolation to get a continuous, almost-everywhere-differentiable curve. Remarkably enough, this converges on the actual solution as Δt shrinks (Arnol'd, 1973).)

Let's try to carry all this over to random functions of time $X(t)$ and $W(t)$. The integral $\int X(t)dt$ is generally not a problem — we just find a version of X

[1] See especially Ethier and Kurtz (1986, Theorem 5.2.12, p. 290).

with measurable sample paths (Section 8.2). $\int X(t)dW$ is also comprehensible if dW/dt exists (almost surely). Unfortunately, we've seen that this is not the case for the Wiener process, which (as you can tell from the W) is what we'd really like to use here. So we can't approximate the integral with a sum like Eq. 18.1. But there's nothing preventing us from using one like Eq. 18.2, since that only demands increments of W. So what we would like to say is that

$$\int_{t=a}^{t=b} X(t)dW \equiv \lim_{n\to\infty} \sum_{i=1}^{n} X\left(t_{i-1}\right)\left(W\left(t_i\right) - W\left(t_{i-1}\right)\right) \tag{18.4}$$

This is a crude-but-workable approach to numerically evaluating stochastic integrals, and apparently how the first stochastic integrals were defined, back in the 1920s. Notice that it is going to make the integral a *random variable*, i.e., a measurable function of ω. Notice also that I haven't said anything yet which should lead you to believe that the limit on the right-hand side exists, in any sense, or that it is independent of the choice of partitions $a = t_0 < t_1 < \ldots t_n\ b$. The next chapter will attempt to rectify this.

(When it comes to the SDE $dX = f(X)dt + g(X)dW$, the counterpart of Eq. 18.3 is

$$X(t + \Delta t) = X(t) + f(X(t))\Delta t + g(X(t))\Delta W \tag{18.5}$$

where $\Delta W = W(t+\Delta t) - W(t)$, and again we use linear interpolation in between the points, starting from $X(t_0) = x_0$.)