The Intersection of Statistics and Topology:

Confidence Sets for Persistence Diagrams

Brittany Terese Fasy

joint work with S. Balakrishnan, F. Lecci,
A. Rinaldo, A. Singh, L. Wasserman

3 June 2014

[**F**LRWBS] Statistical Inference for Persistent Homology: Confidence Sets for Persistence Diagrams. ArXiv 1303.7117. Tentatively accepted, Annals of Statistics.

The Intersection of Statistics and Topology:

Stochastic Convergence of Persistence Landscapes and Silhouettes

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3 June 2014

[C**F**LRW] Stochastic Convergence of Persistence Landscapes and Silhouettes. ArXiv. SoCG 2014 Proceedings, Kyoto. Journal version pending.

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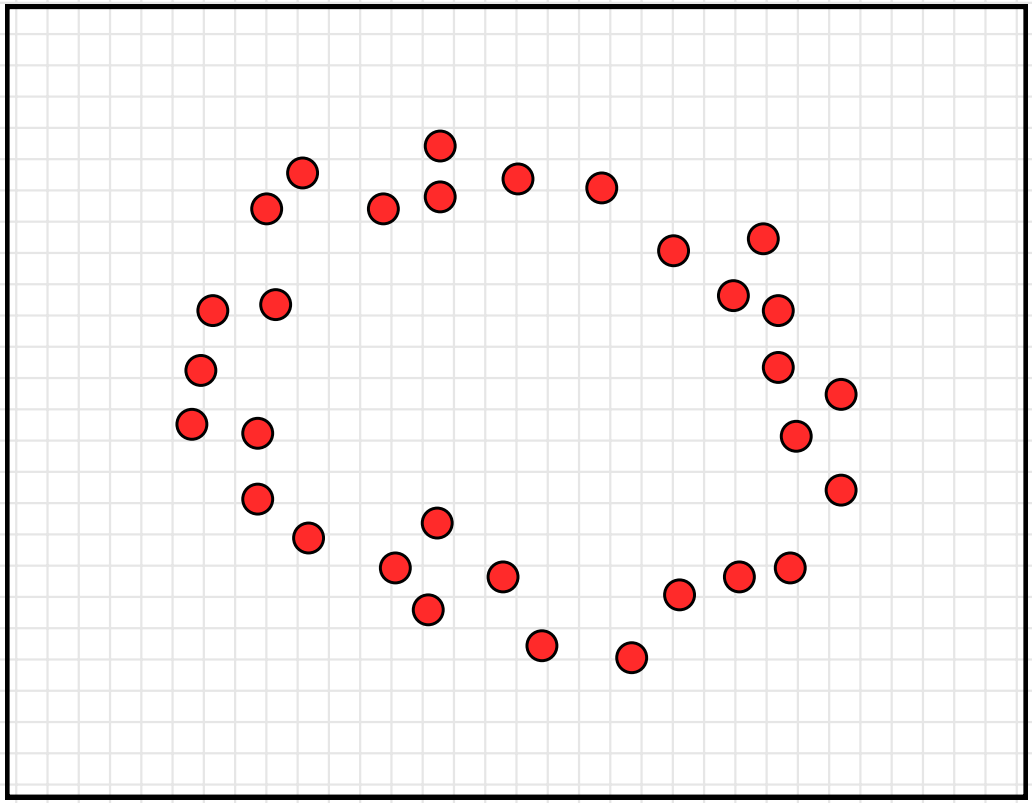
Motivation

Extracting Structure from Point Cloud Data



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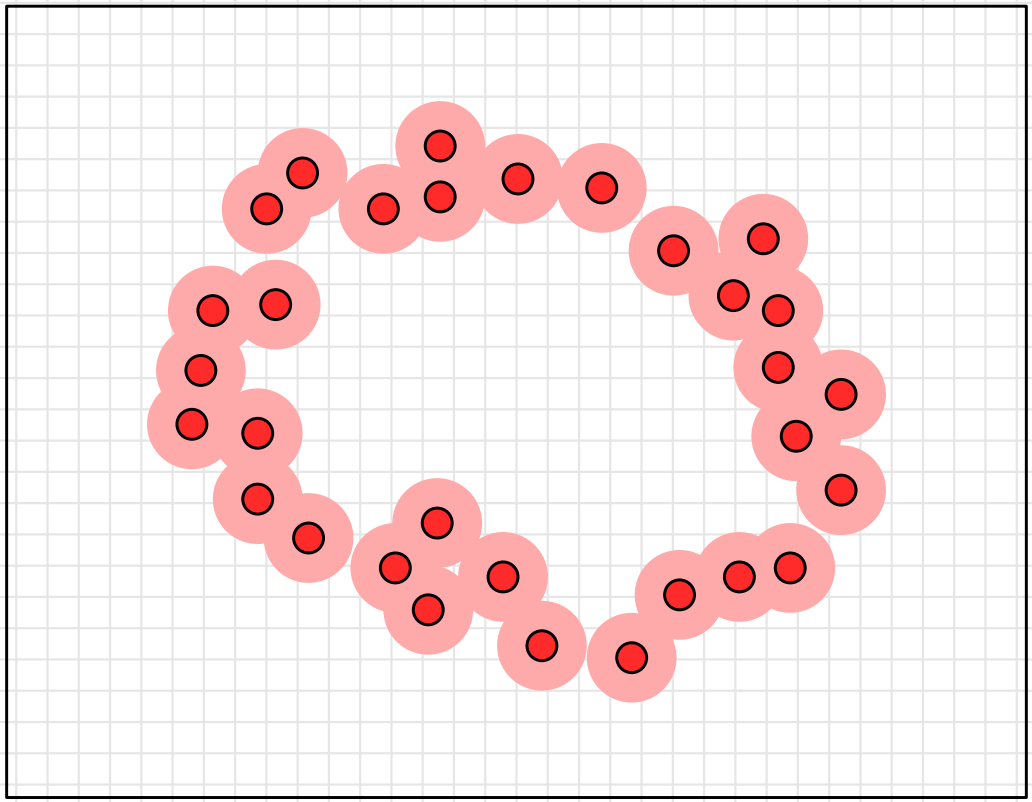

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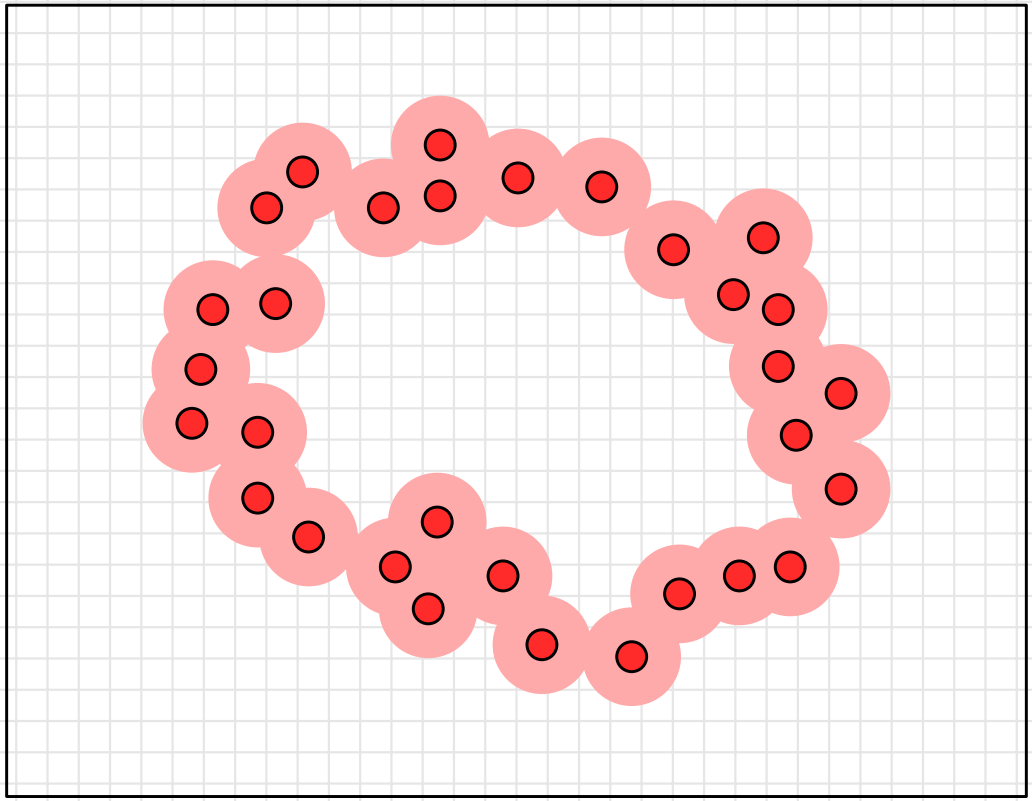

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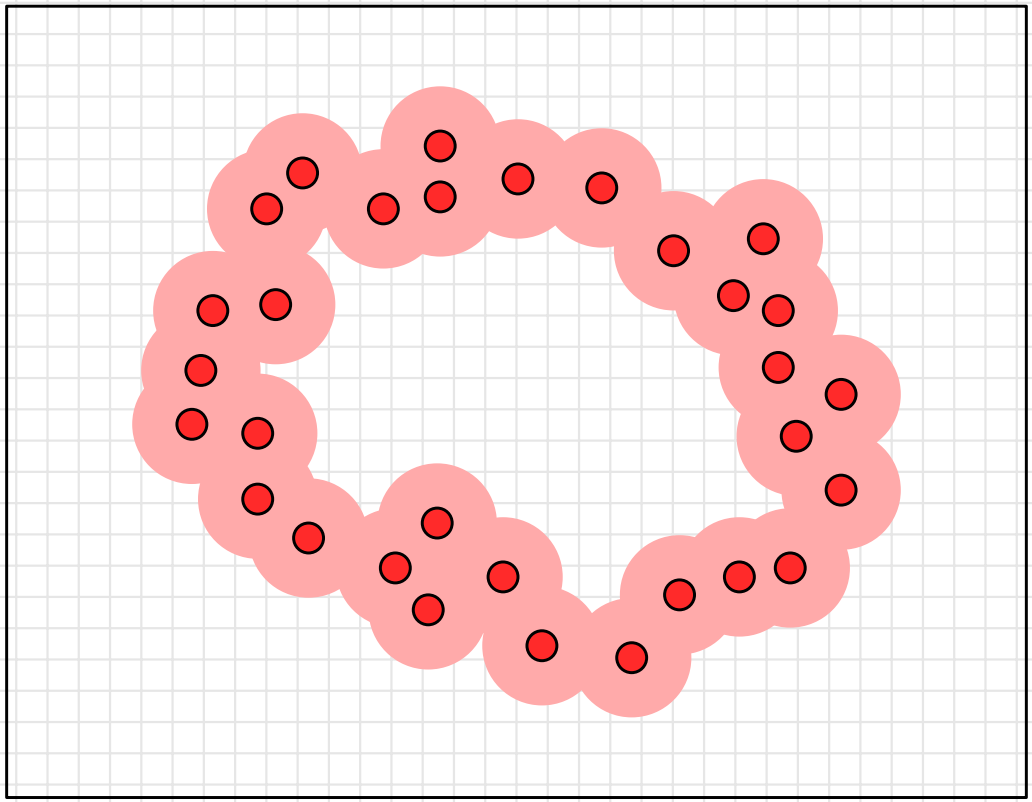

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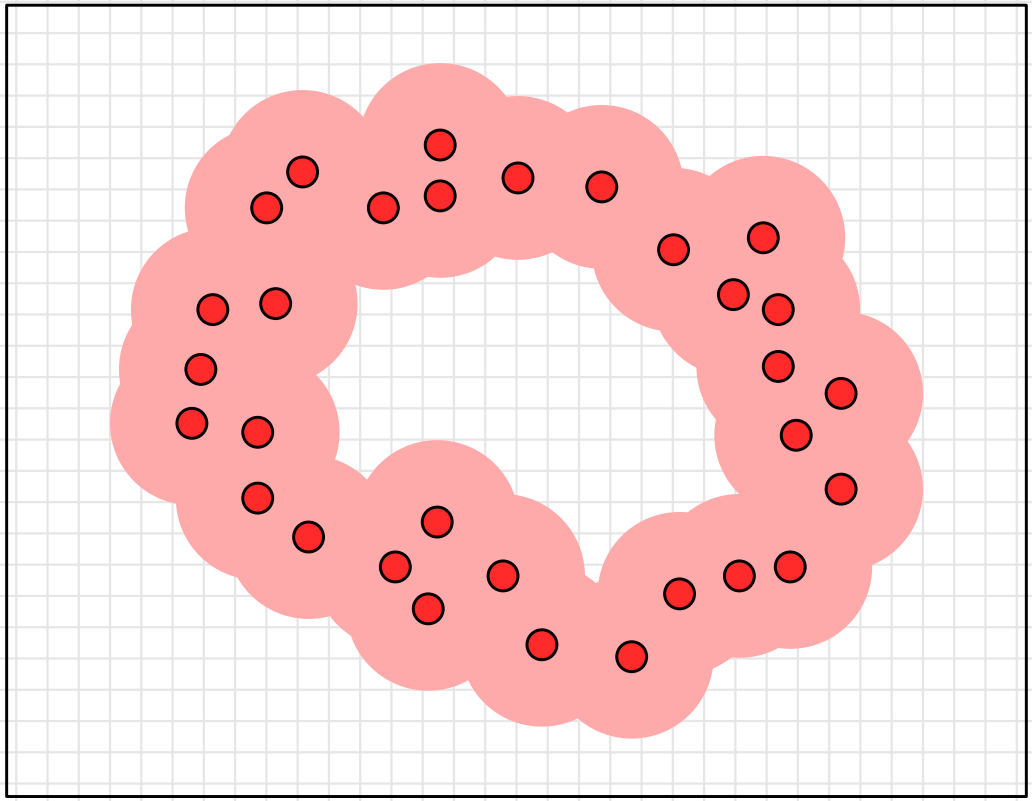

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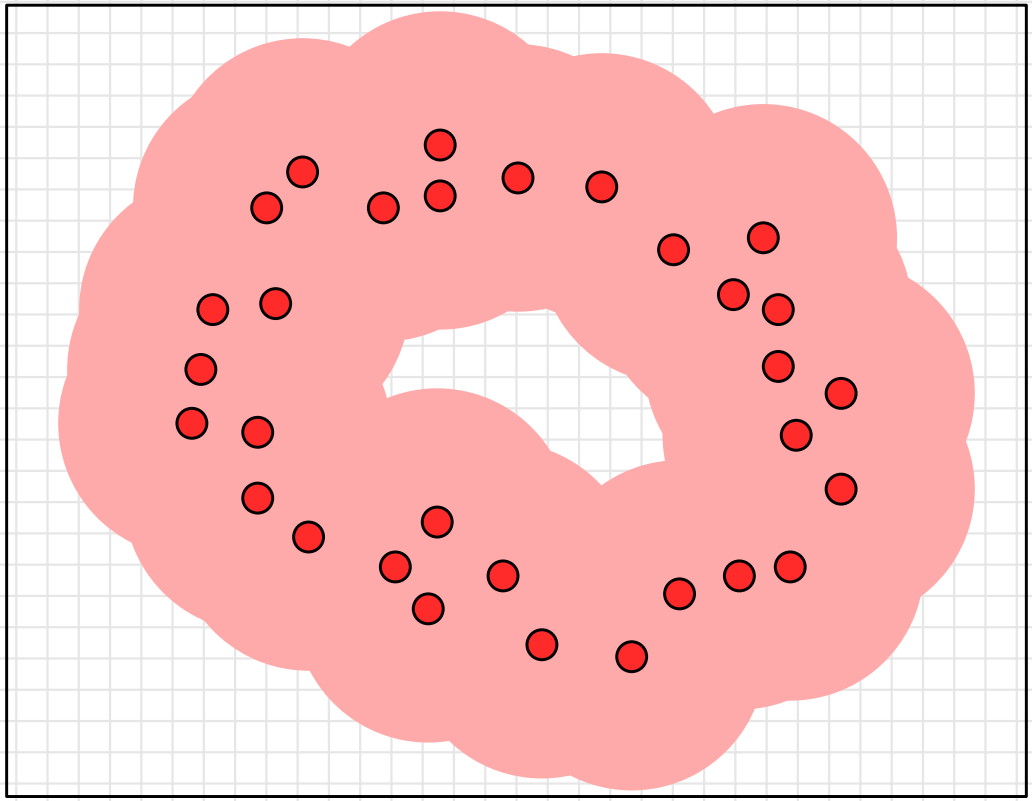

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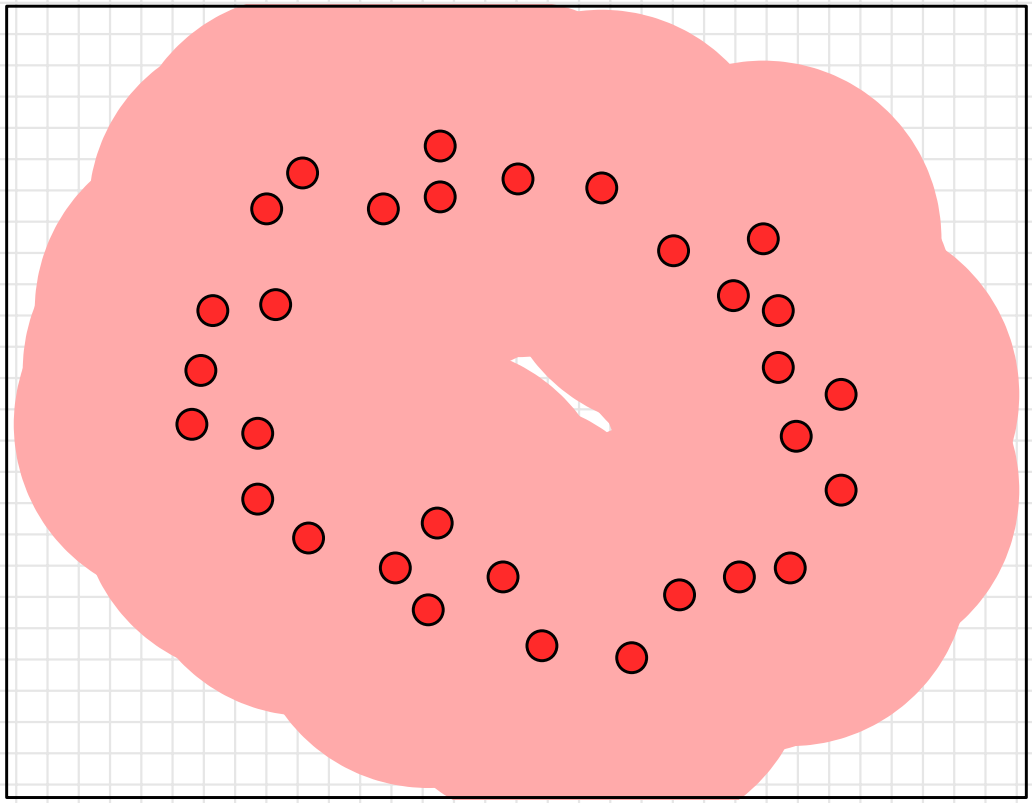

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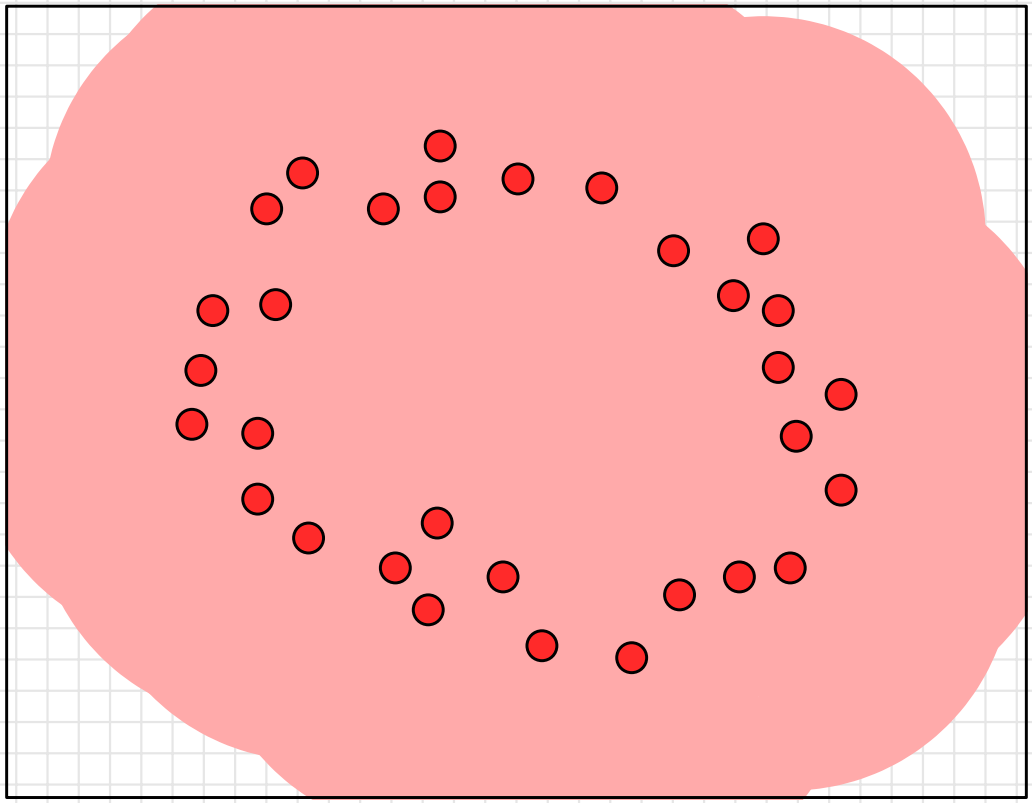

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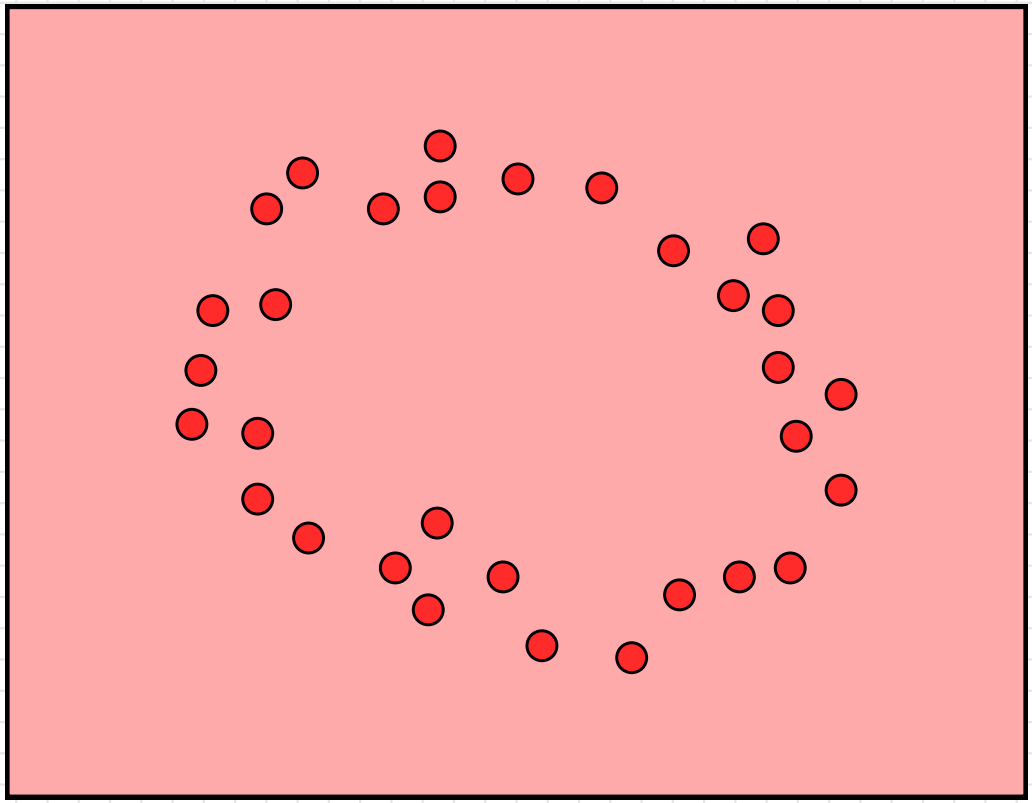

Homeomorphisms

An Informal Discussion

A *homeomorphism* is a continuous bending, stretching, or shrinking (but not tearing or glueing) of one object into another object.

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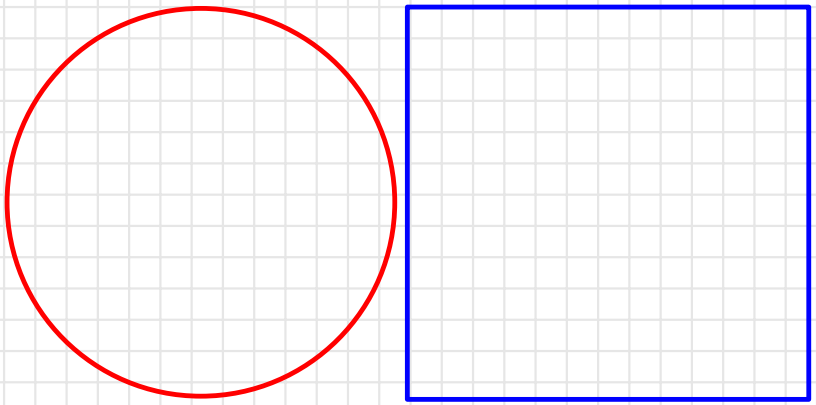

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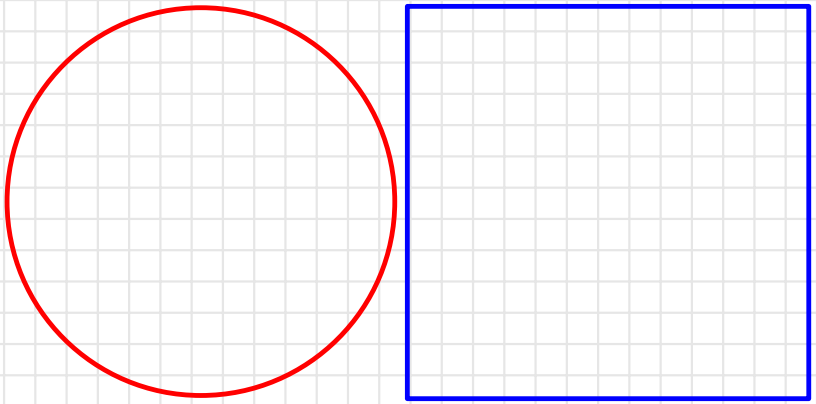

Donut = Coffee Cup

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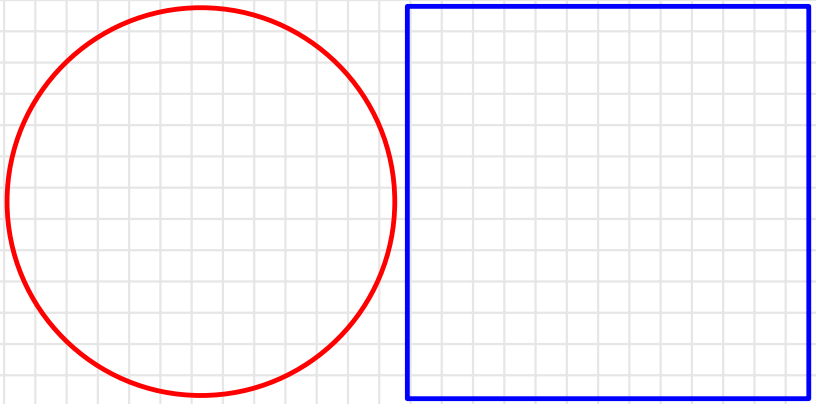
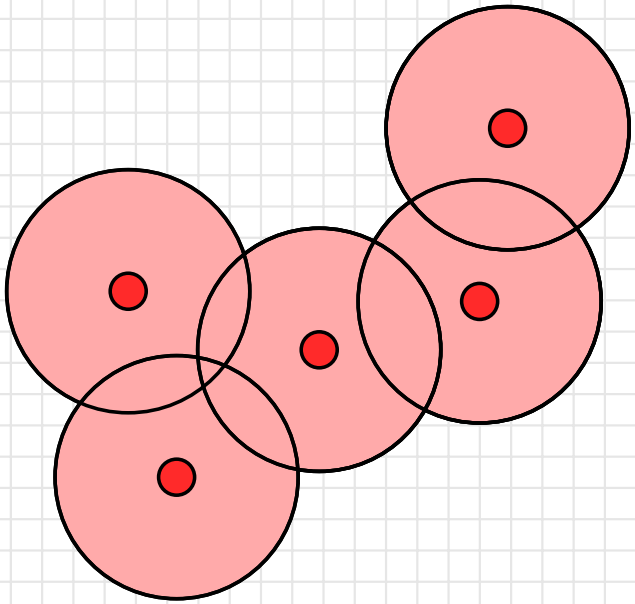


Donut ≠ Sugar Bowl

Topological Spaces

Nerve Complexes



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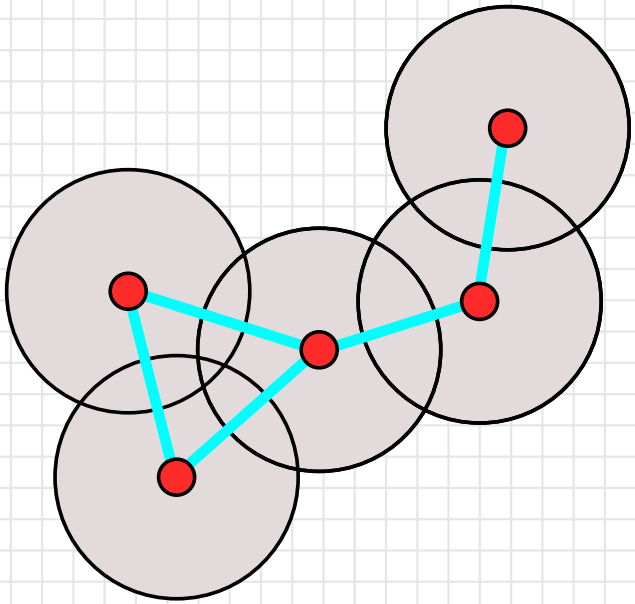

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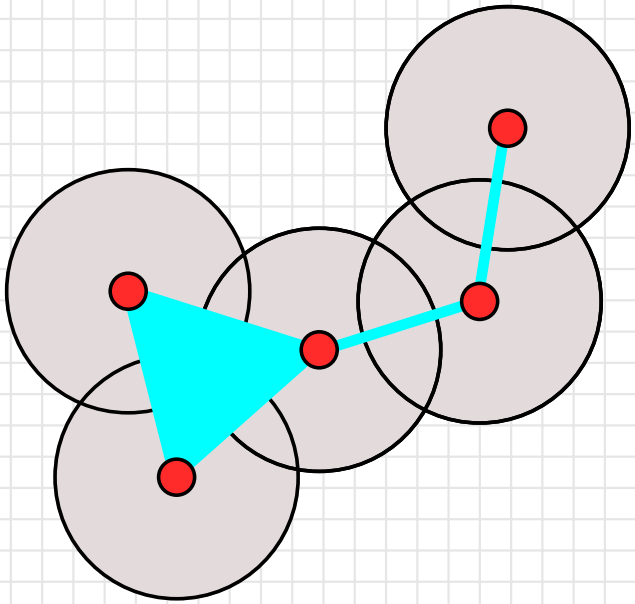

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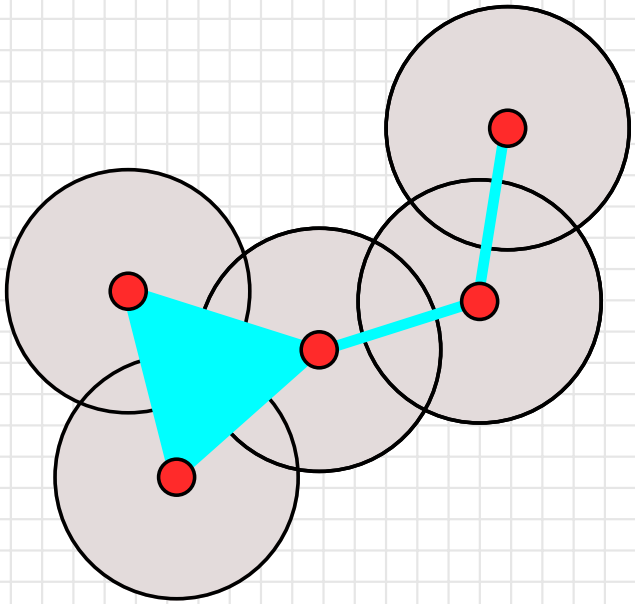
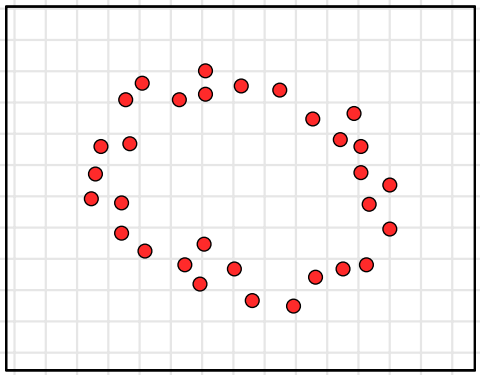


Nerve Lemma

If X is a set of closed convex sets, then $\mathrm{Nrv}(X)$ is topologically equivalent to $\bigcup X$ (up to homotopy type).

Homology



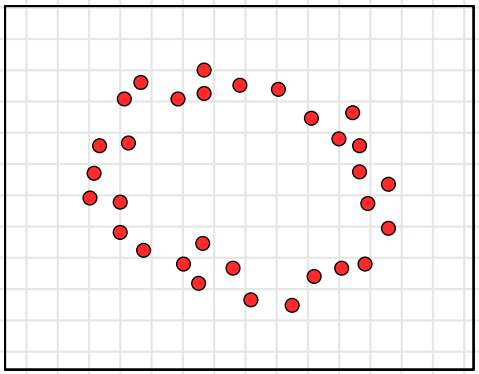
$\beta_0 = n$

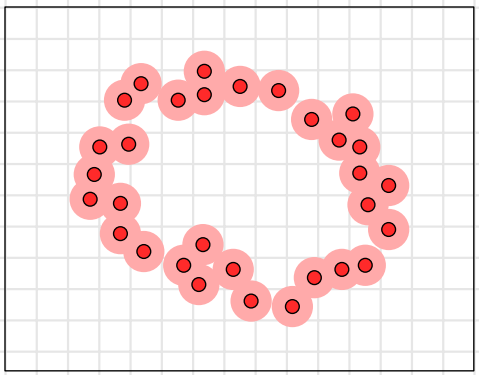
$\beta_1 = 0$

Simplicial Homology

- X_p is the power set of all p-simplices.
- $C_p = (X_p, +_2)$.
- $\partial_p : C_p \to C_{p-1}$ is the boundary map.
- $H_p = \mathrm{Ker}(\partial_p)/\mathrm{Im}(\partial_{p+1})$
- $\beta_p = \mathrm{rank}(H_p)$.

Homology





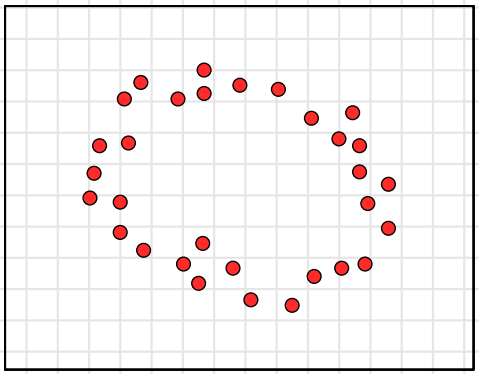
$\beta_0 = n$

$\beta_1 = 0$

$\beta_0 = 5$

$\beta_1 = 1$

Homology



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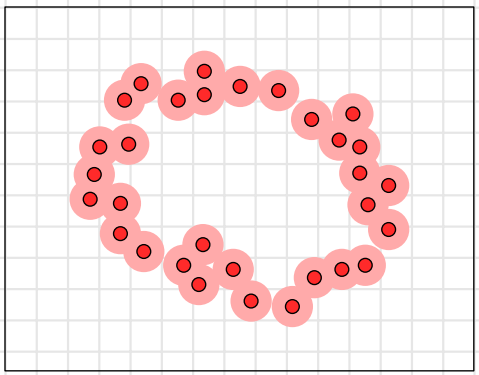

$\beta_0 = 5$

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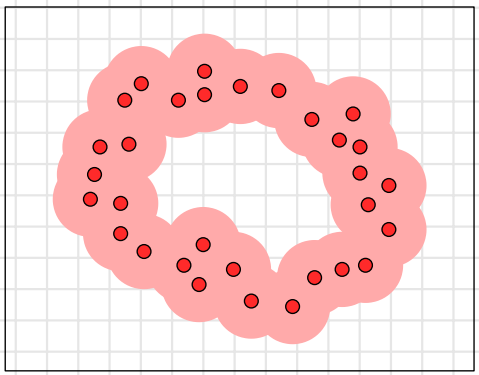

$\beta_0 = 1$

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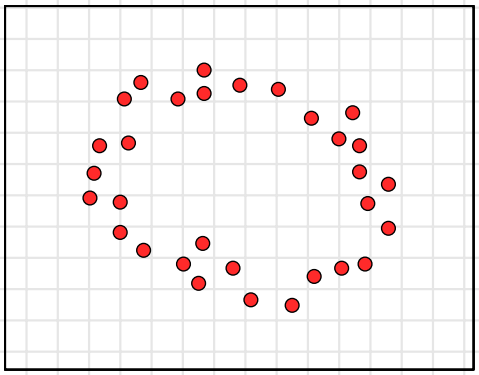
Homology



$\beta_0 = n$

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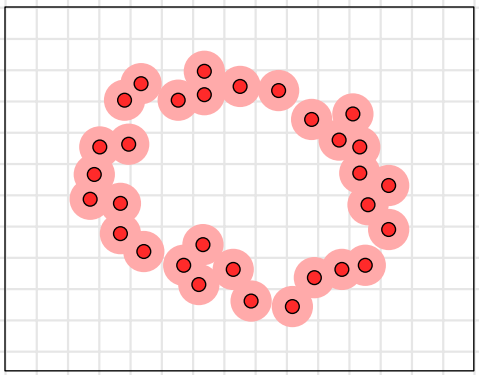

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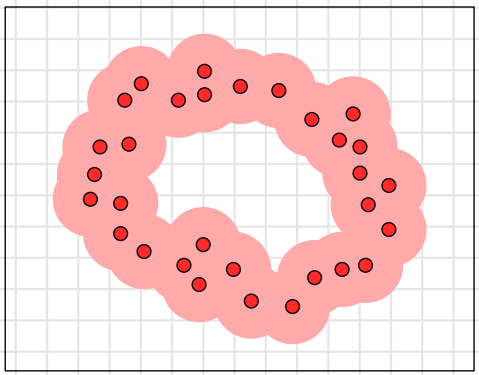

$\beta_0 = 1$

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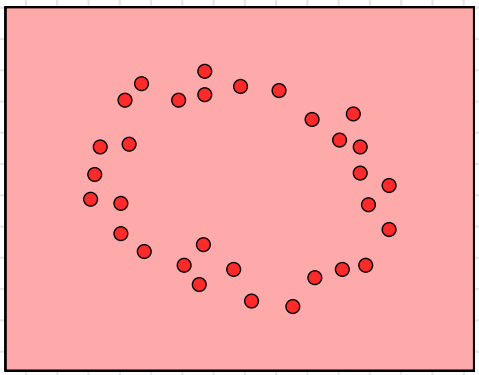

$\beta_0 = 1$

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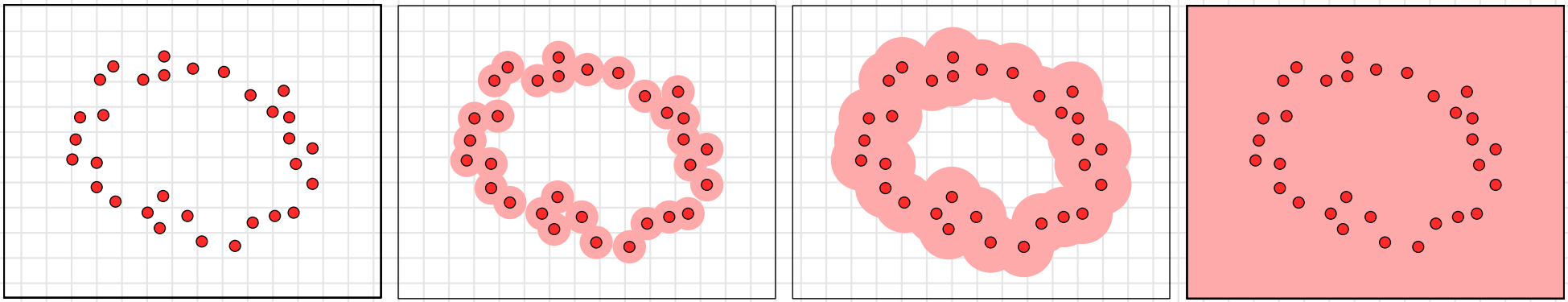
Homology

$d^{-1}(0)$ $d^{-1}([0, r_1])$ $d^{-1}([0, r_2])$ $d^{-1}([0, r_4])$



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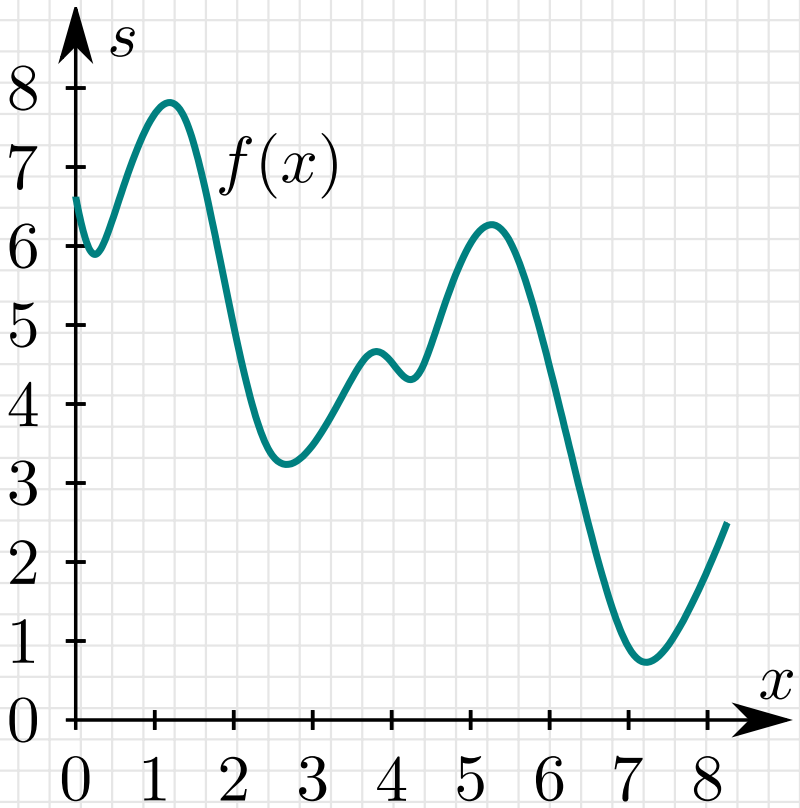
Persistent Homology

Lower-Level Set Filtration

Let $\mathbb{M}$ be a compact topological space.
Let $f : \mathbb{M} \to \mathbb{R}$ be continuous.
We consider all sets of the form $\mathbb{M}_t := f^{-1}((-\infty, t])$.



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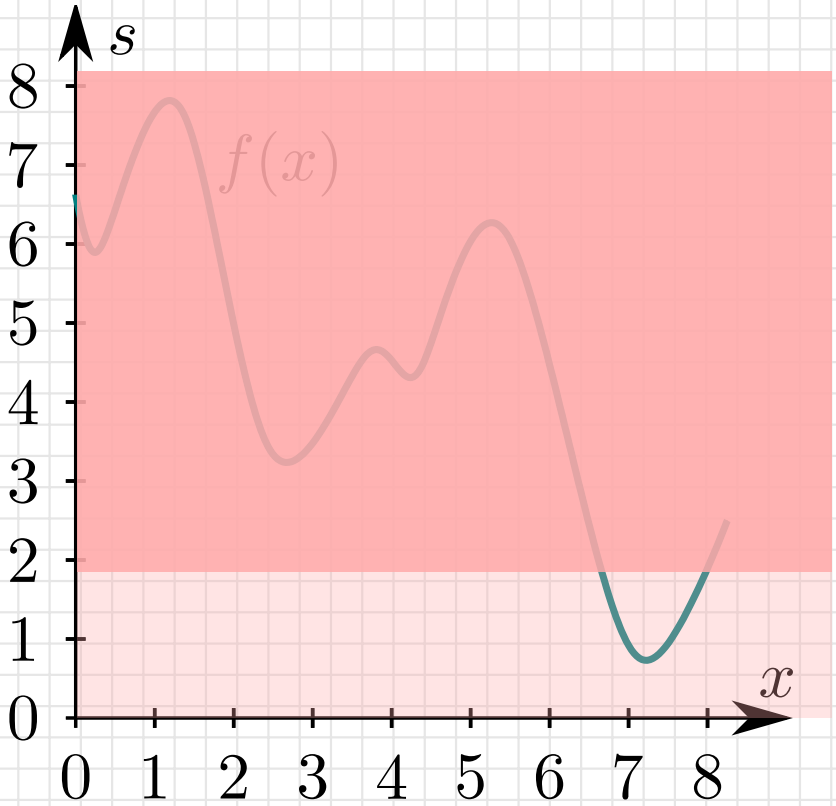

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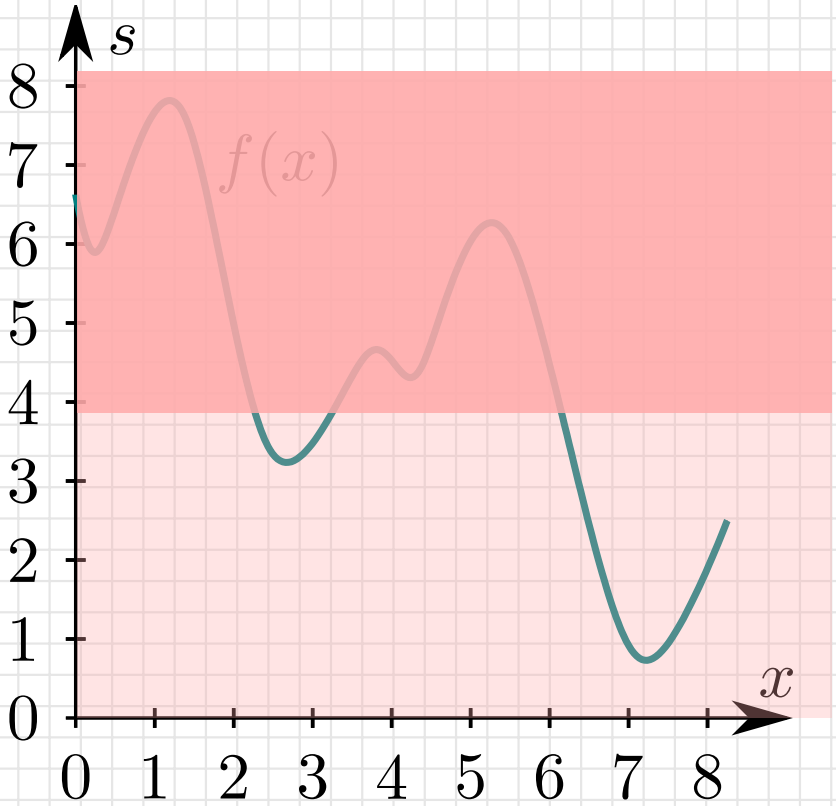

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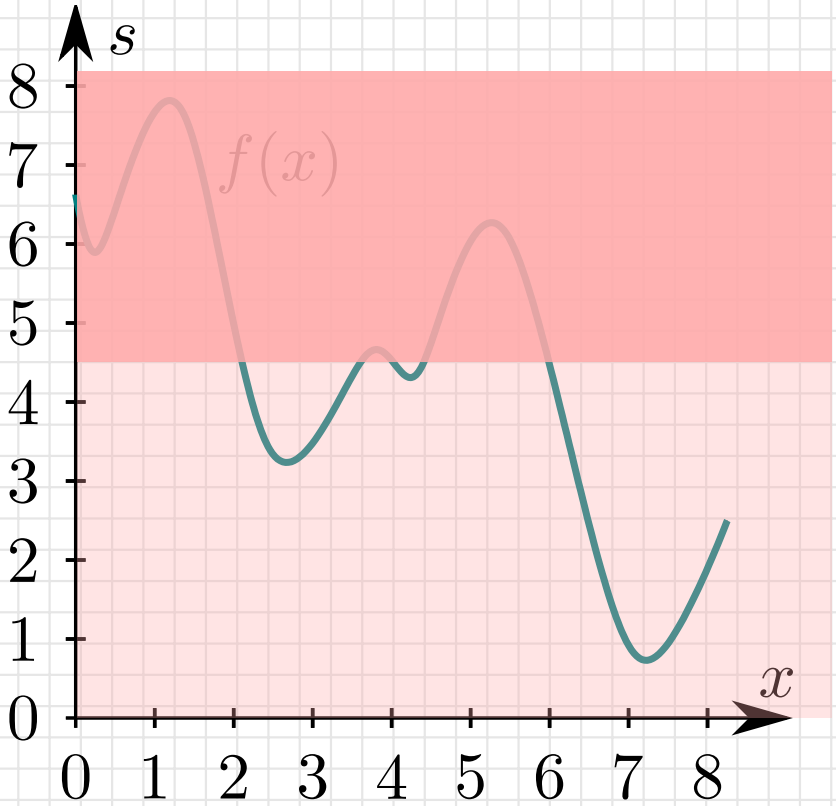

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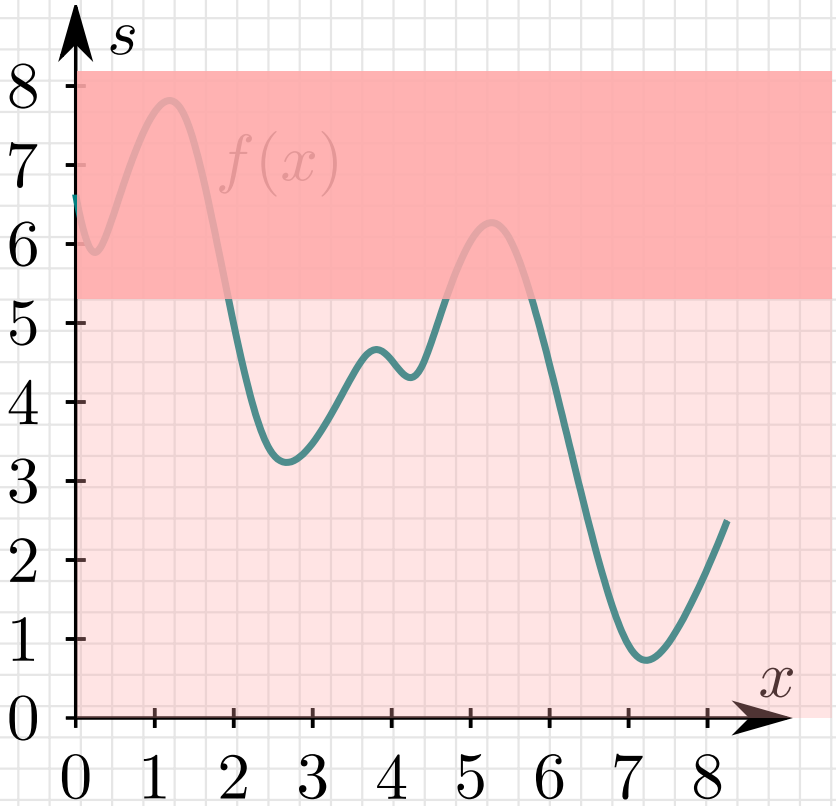

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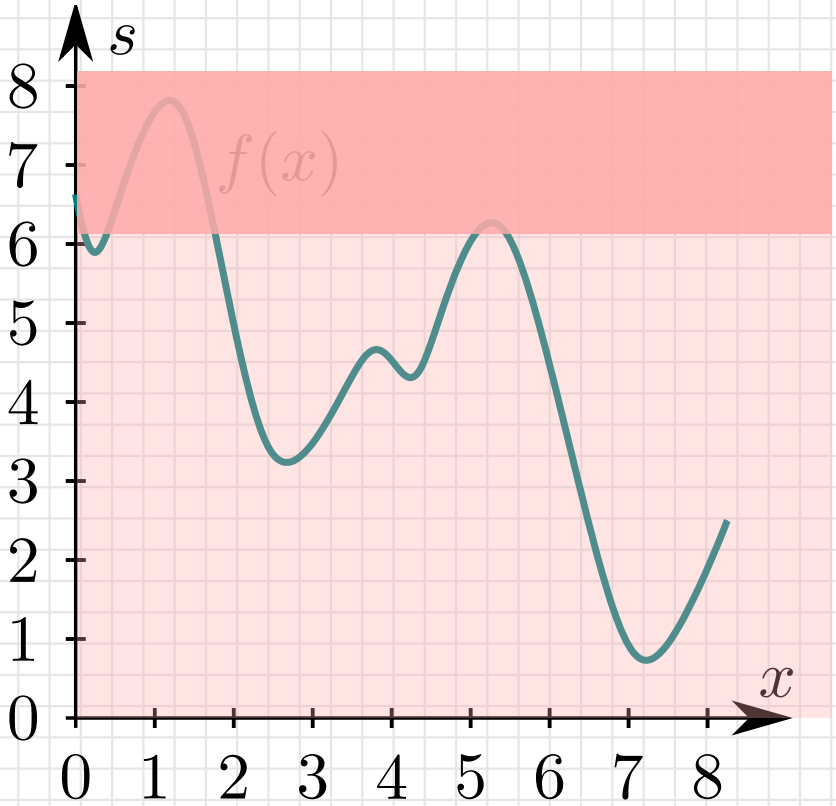

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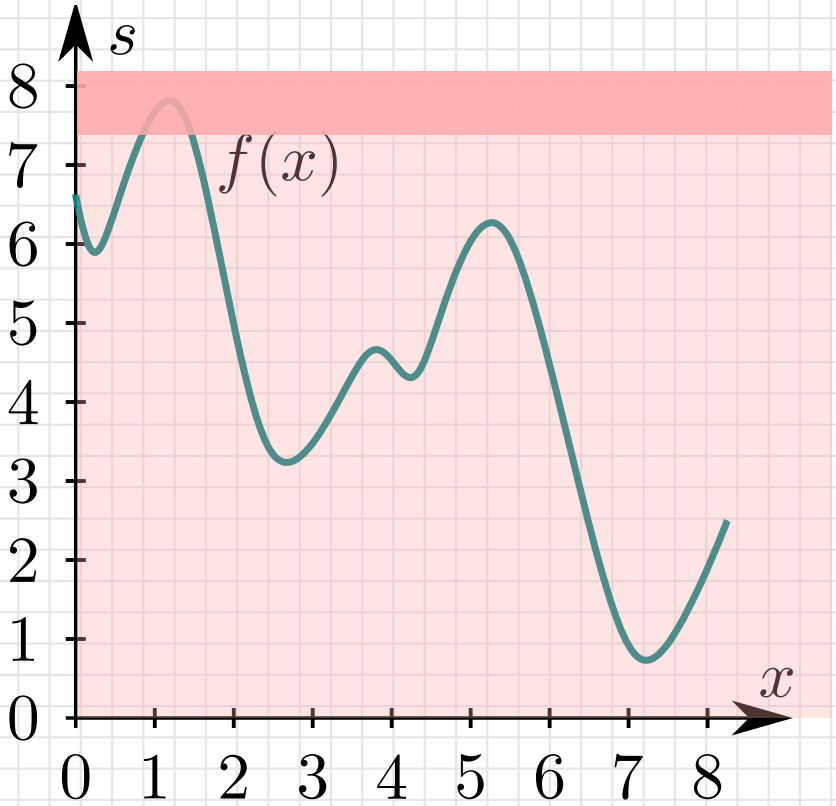

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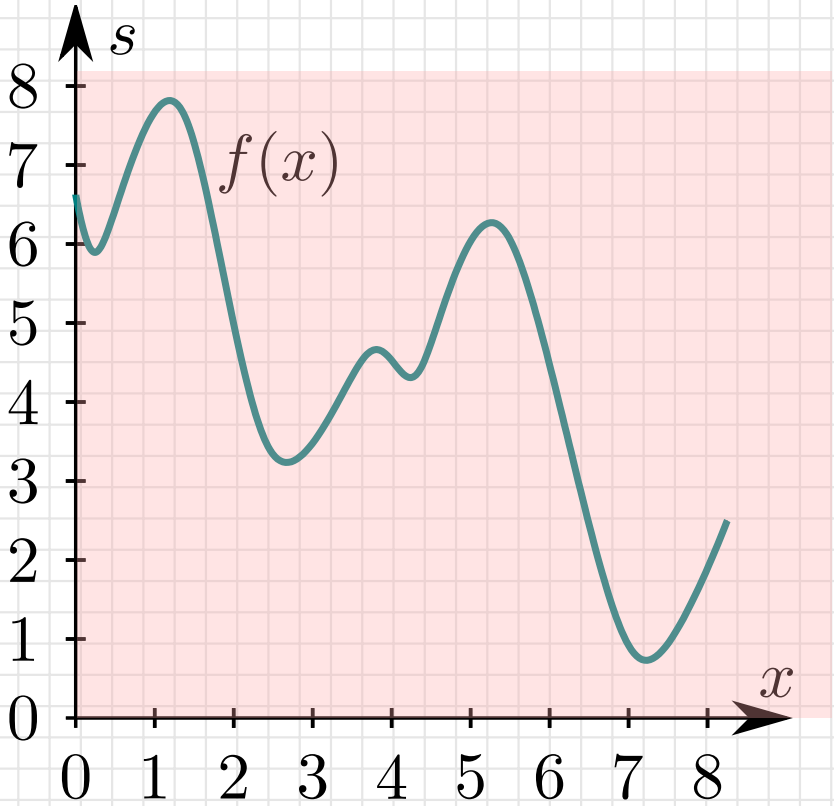

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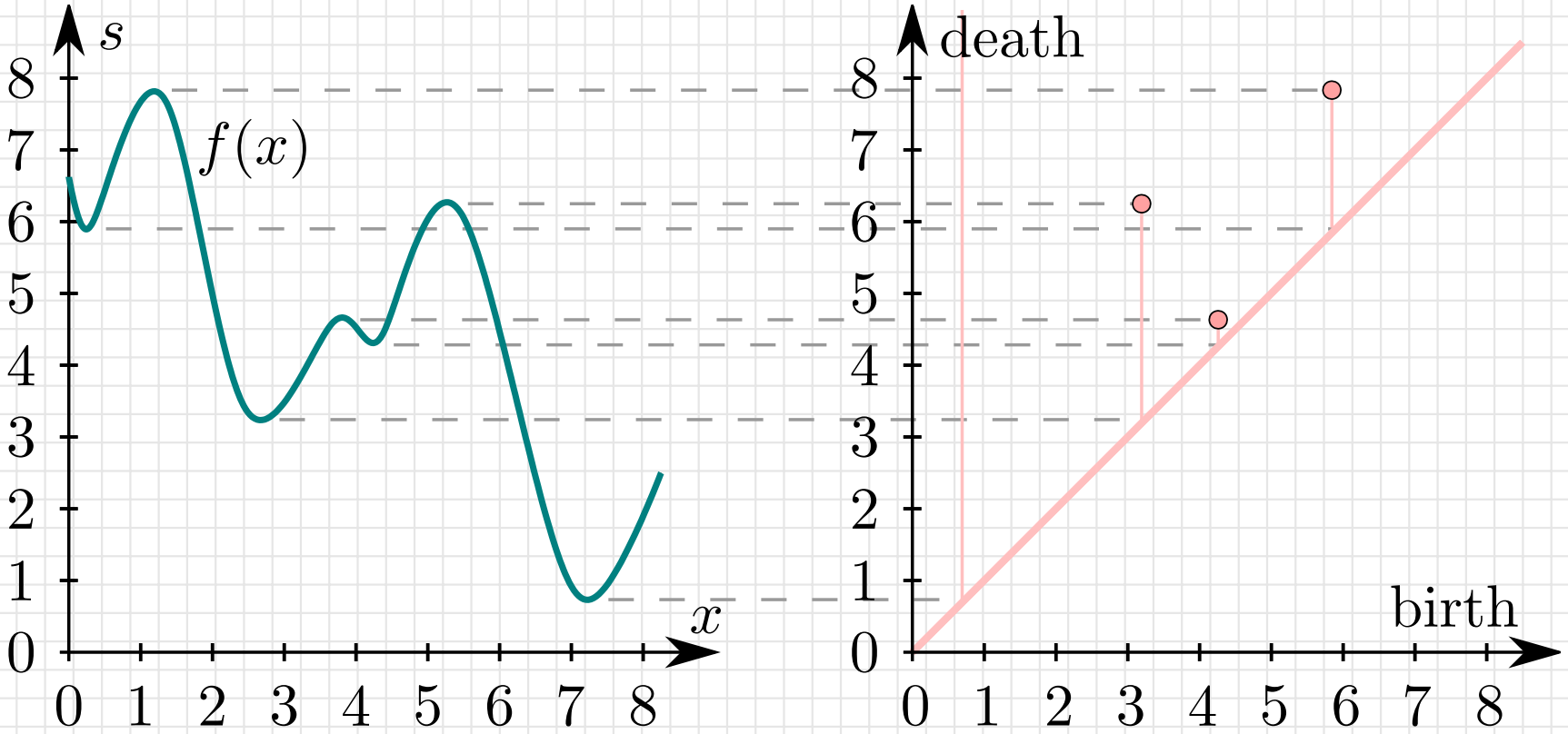

Persistent Homology

Upper-Level Set Filtration

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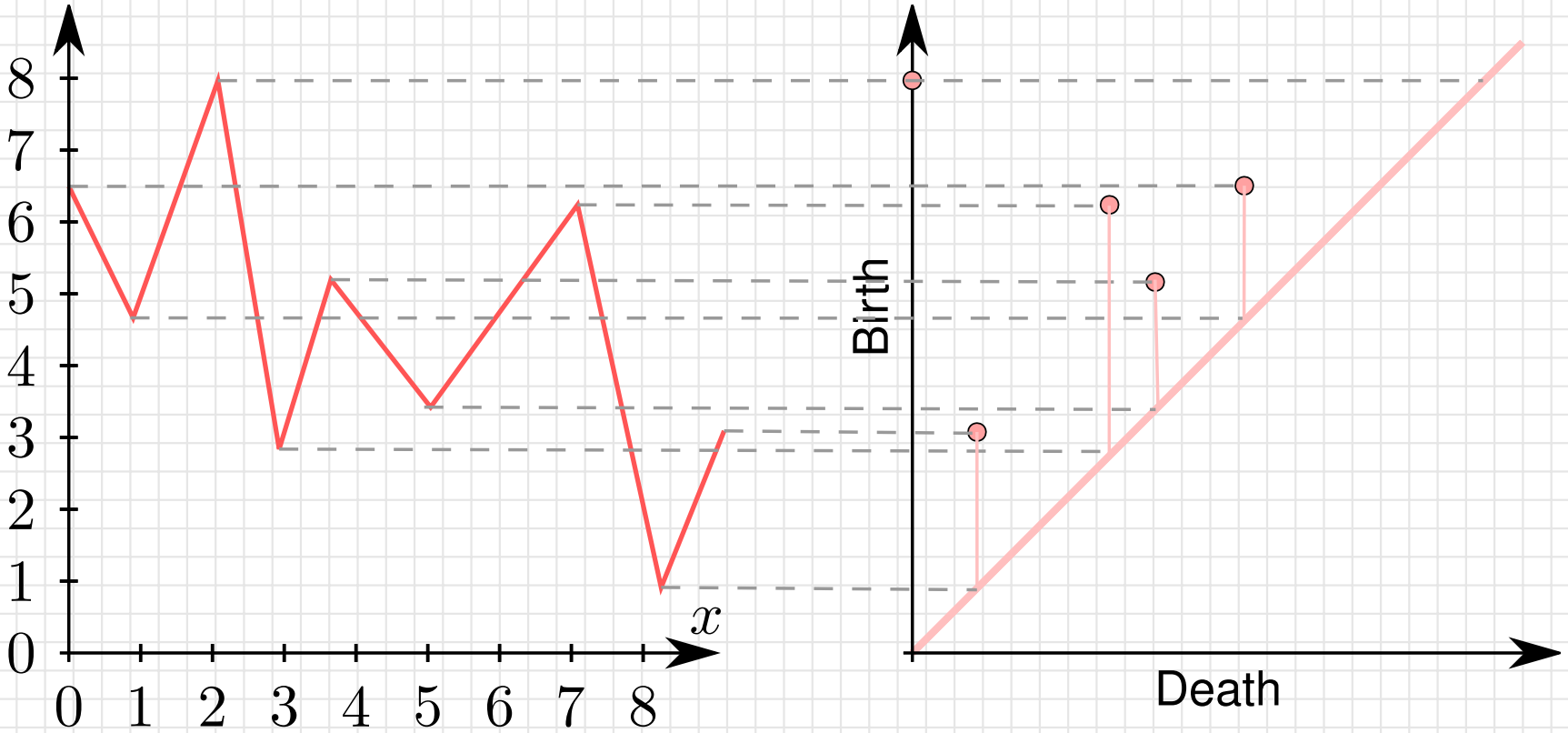
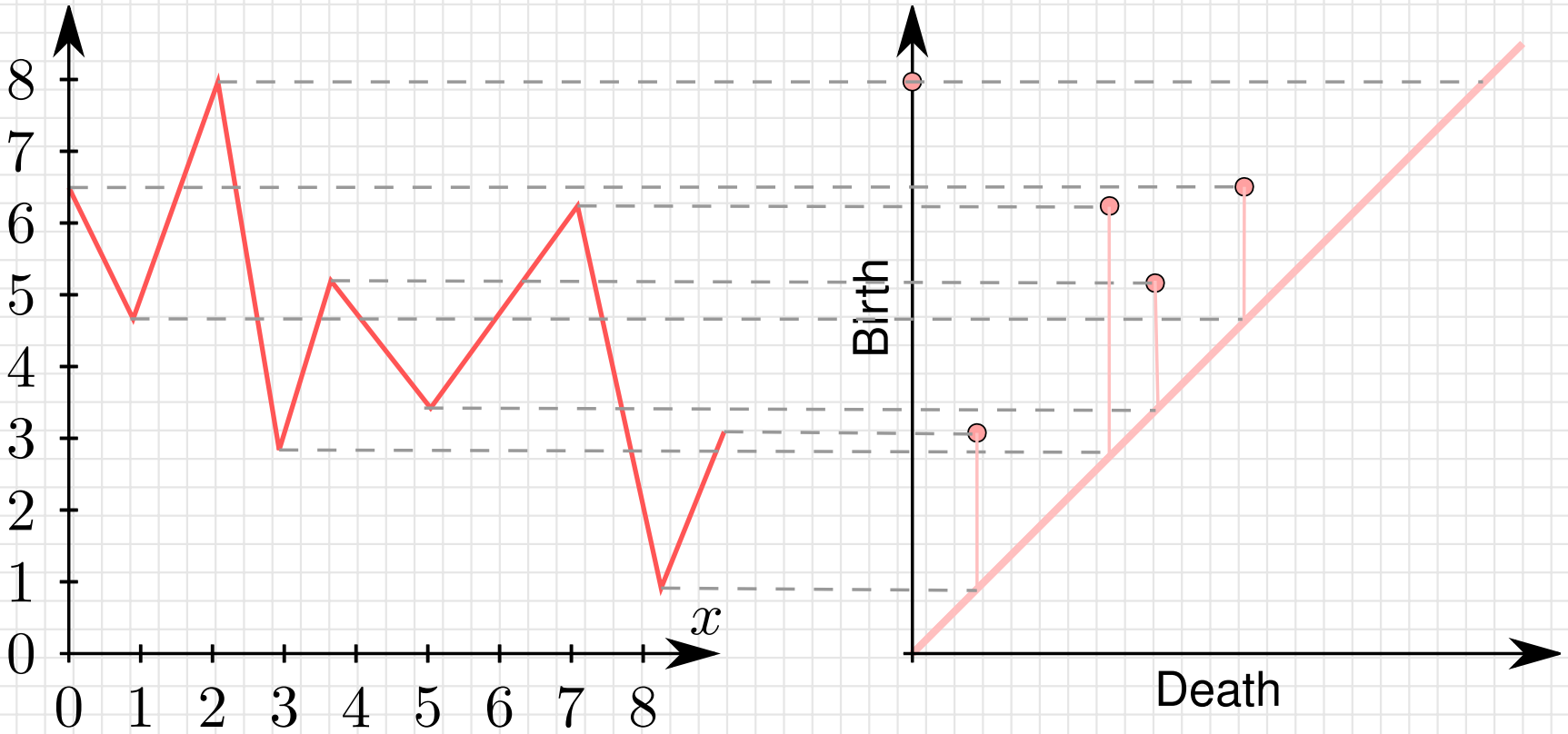


Distance and Stability

A Metric on Persistence Diagrams



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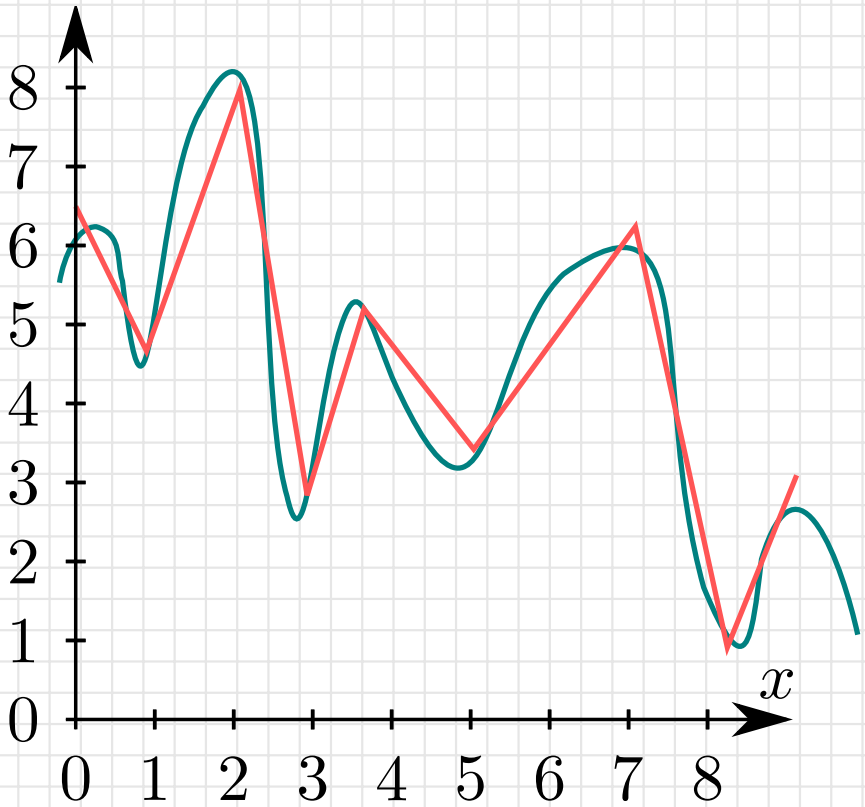
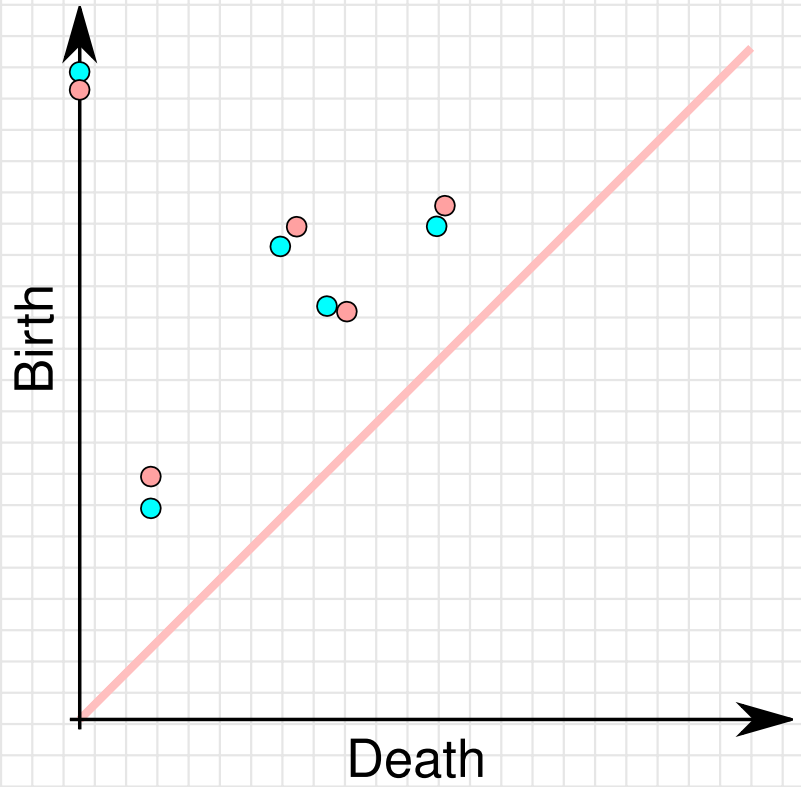




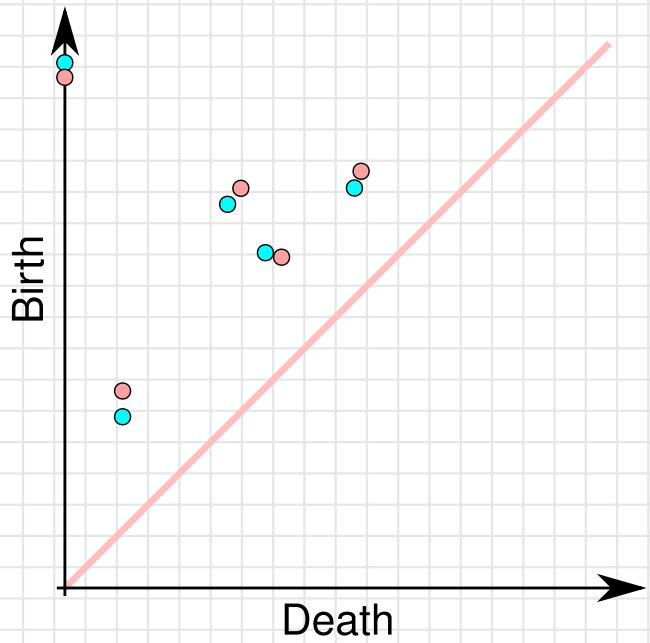
Bottleneck Distance

Given two persistence diagrams, find the best *perfect matching* M between the point sets.

Minimize Cost

We wish to find

$$W_\infty = \min_M \{ \max_{(p,q)\in M} ||p-q||_\infty \}.$$



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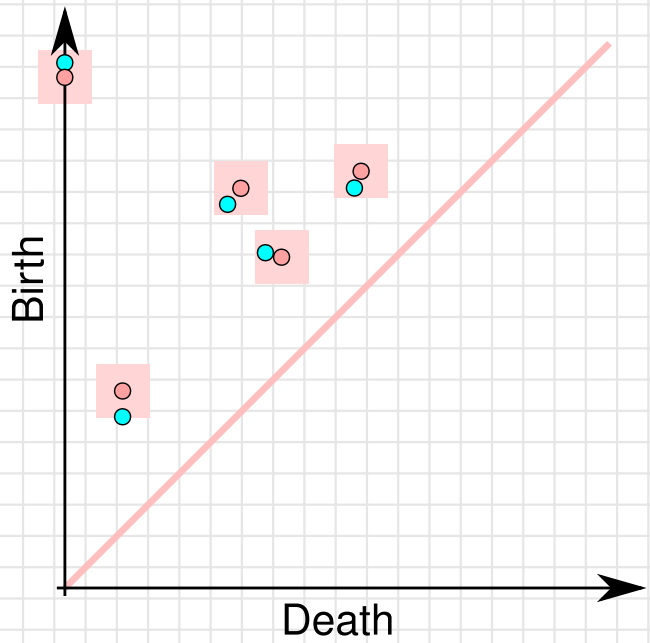
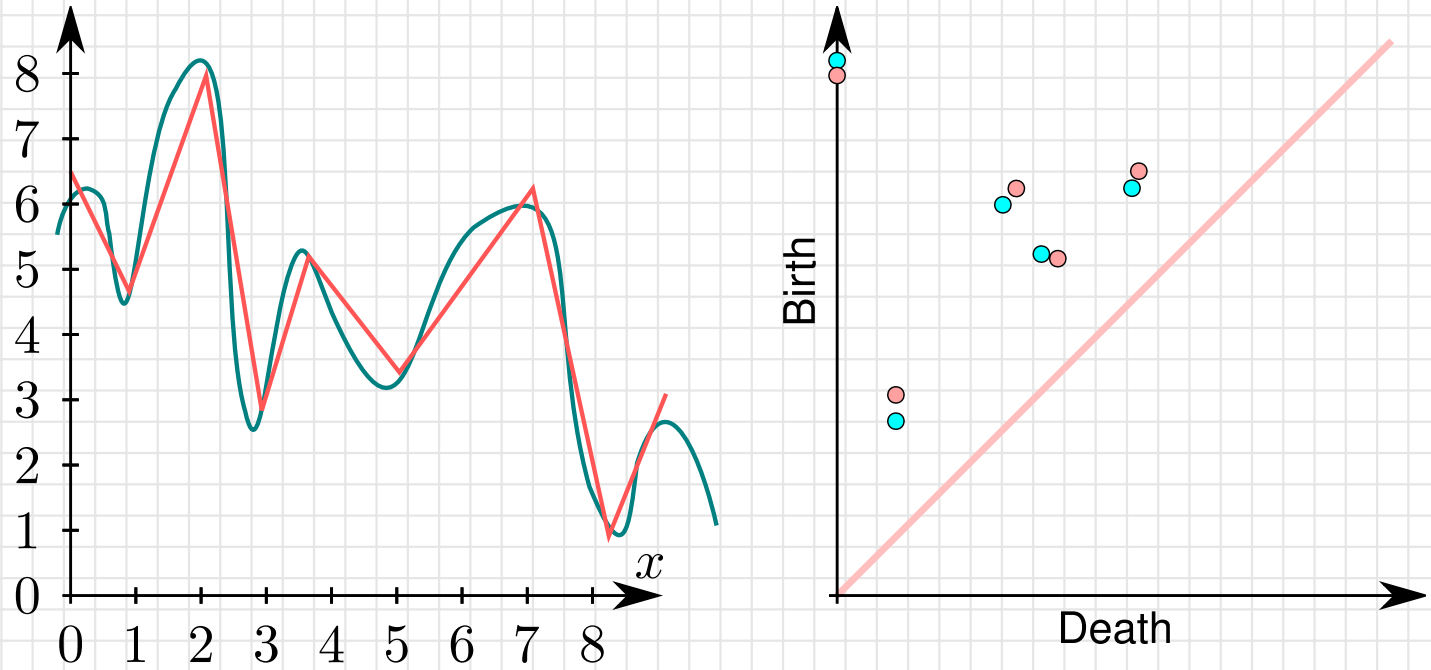


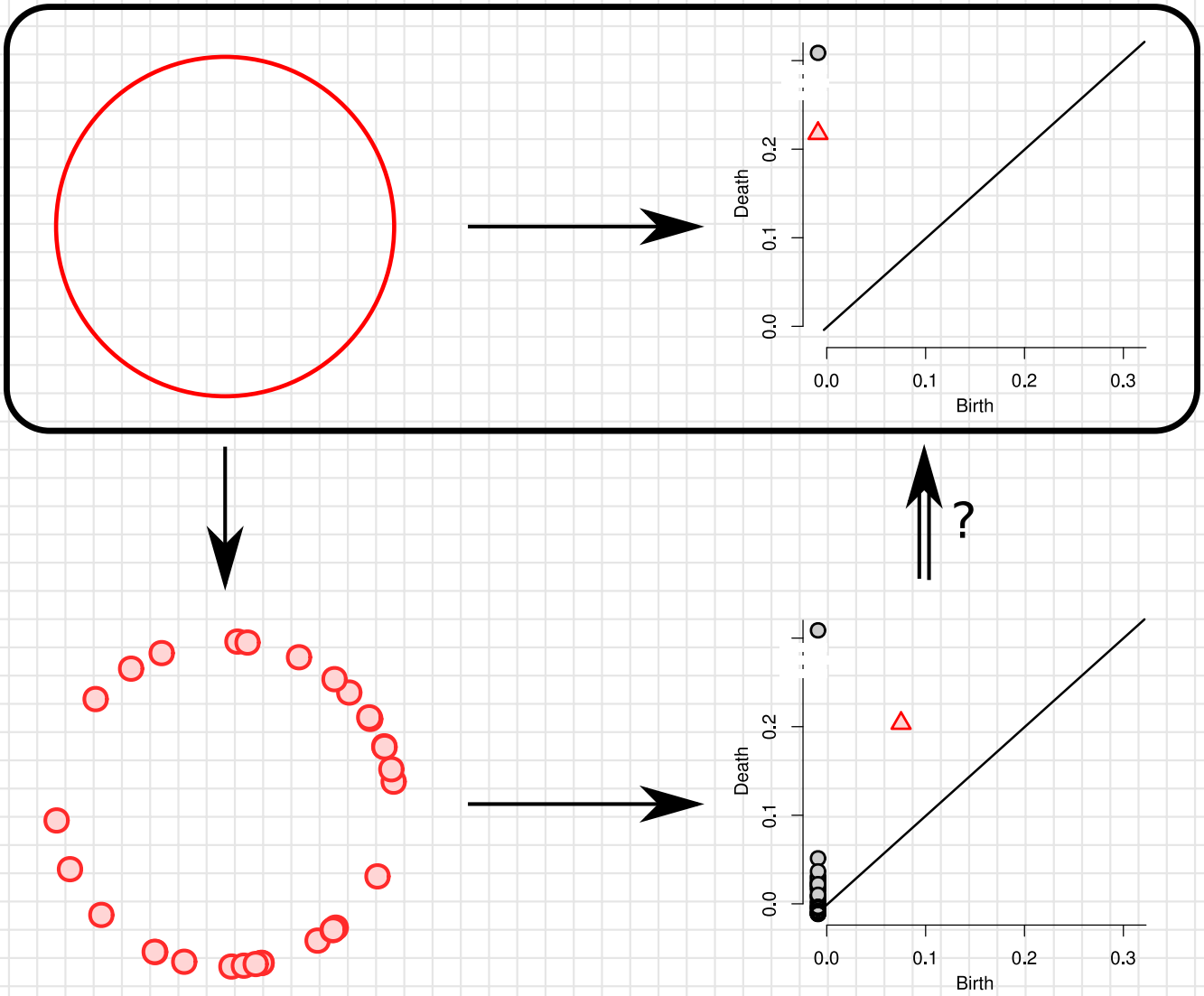
Stability of Matchings



Bottleneck Stability Theorem [CDGO]

$$||f_1 - f_2||_\infty \quad \geq \quad W_\infty(D_1, D_2)$$

Topological Inference



Objective

Let $\mathcal{D}_T$ denote the set of all T-bounded persistence diagrams.

Confidence Sets

Given $\alpha \in (0,1)$ and unknown diagram D, we want $\mathcal{C}_\alpha \subset \mathcal{D}_T$ such that

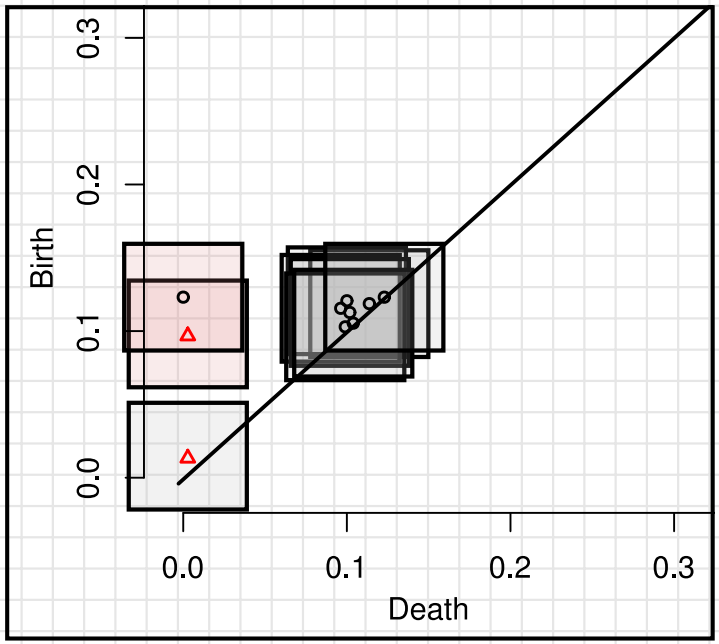
$$\mathbb{P}(D \in \mathcal{C}_\alpha) \geq 1 - \alpha.$$

Question

If $\widehat{D}$ is an estimate of D, how close is $\widehat{D}$ to D ?

Confidence Sets for Persistent Diagrams

$$C_\alpha = \{D \in \mathcal{D}_T \ : \ W_\infty(D, \hat{D}) \leq t(\alpha)\}$$



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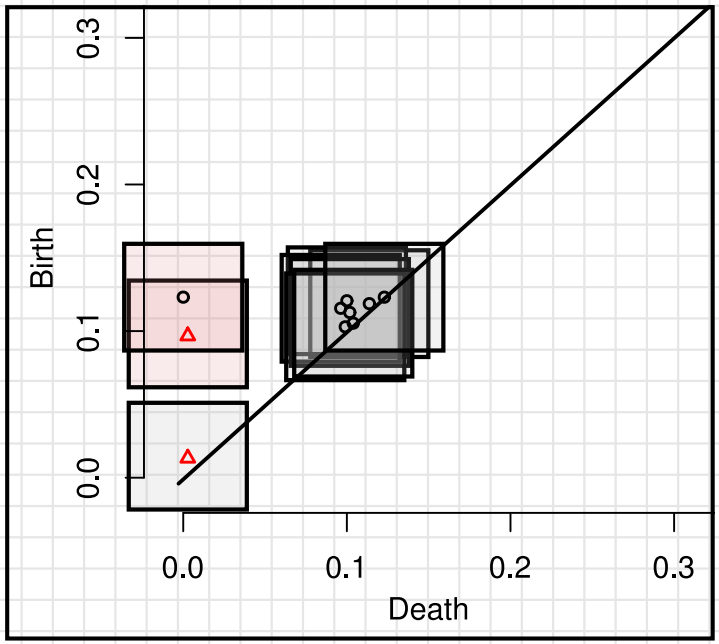

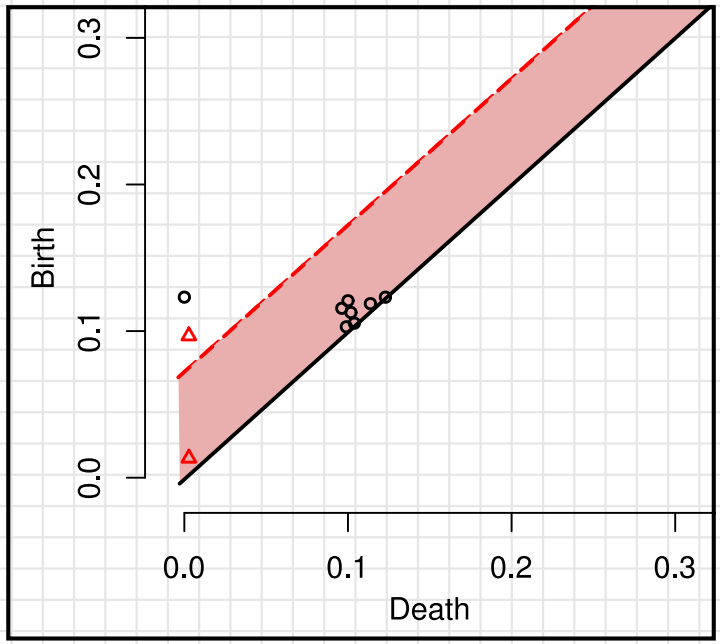

Statistical Model

$\mathbb{M}$ is a manifold.
P is a probability distribution supported on $\mathbb{M}$.
Observe data $X_1, X_2, \ldots, X_n \sim P$.
Compute $\widehat{D}_n = \mathrm{Dgm}^-(X_1, \ldots, X_n)$

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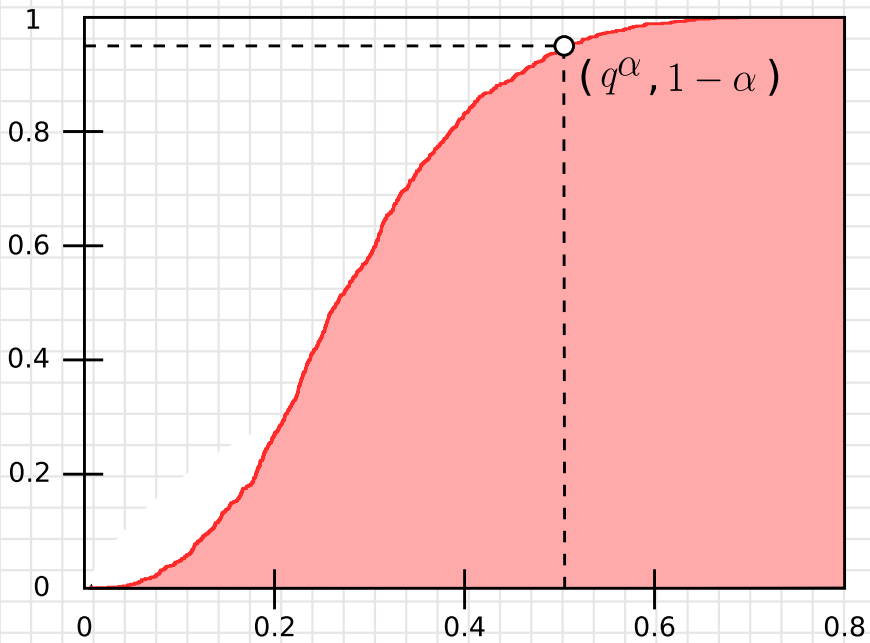
Computing a Confidence Interval

With Infinite Resources

Repeatedly sample n data points, obtaining:



Confidence Intervals

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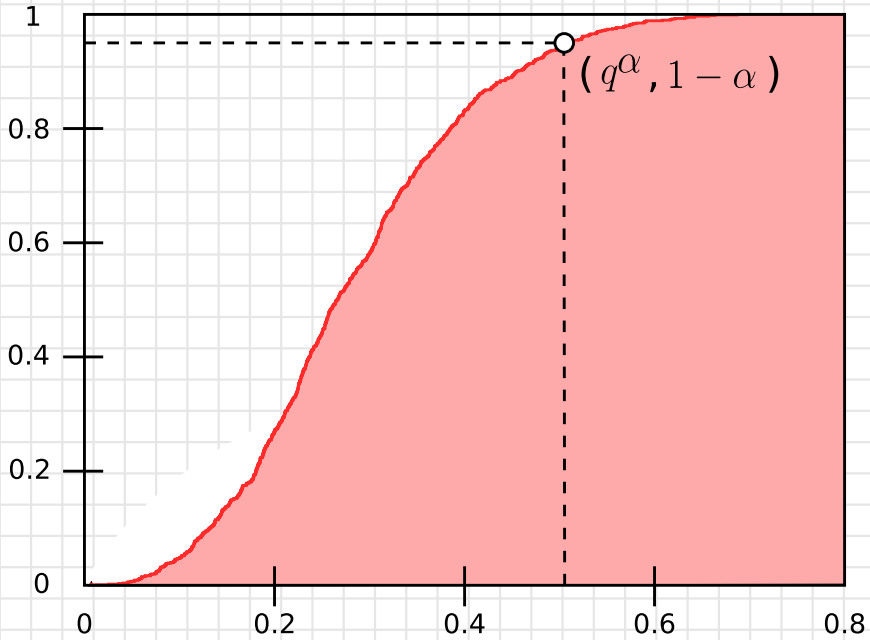

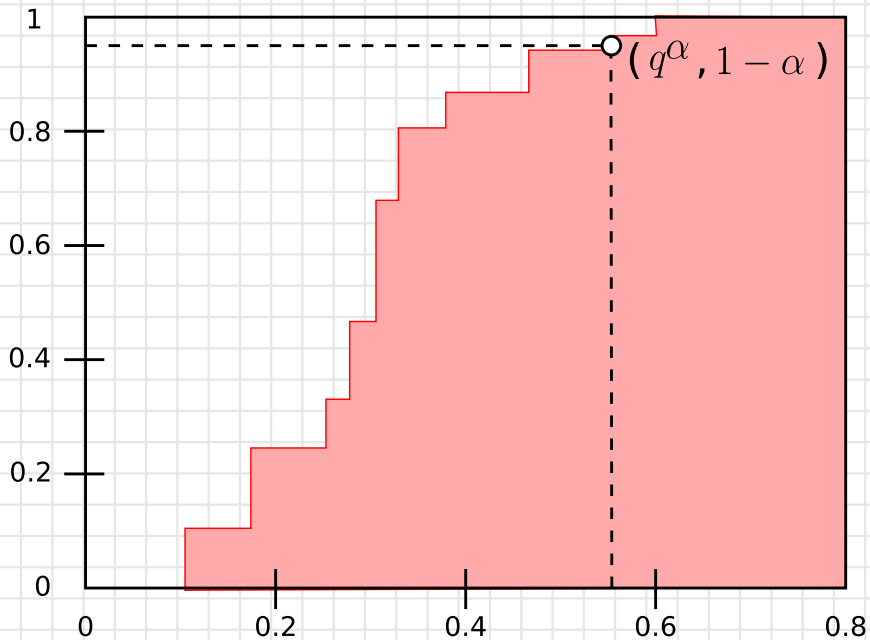

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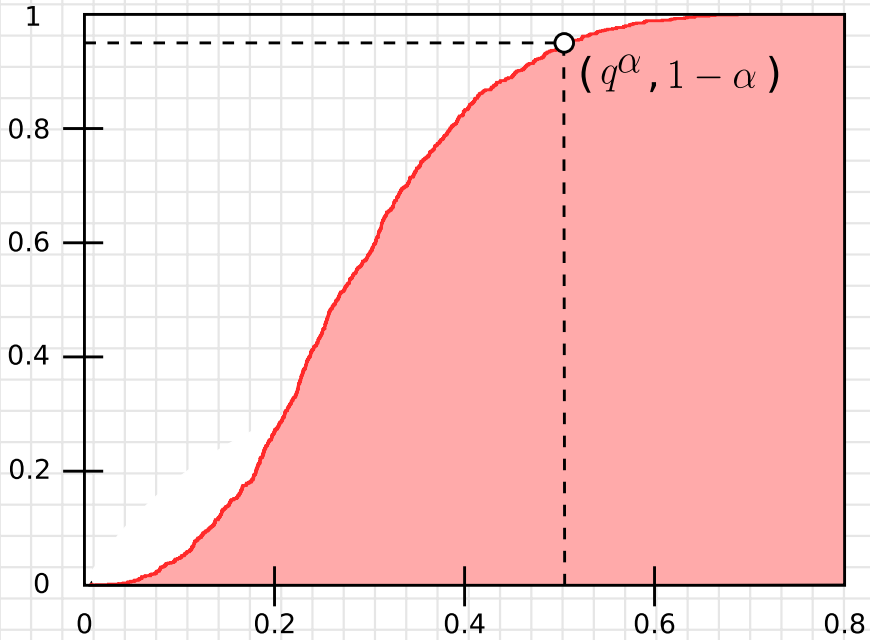

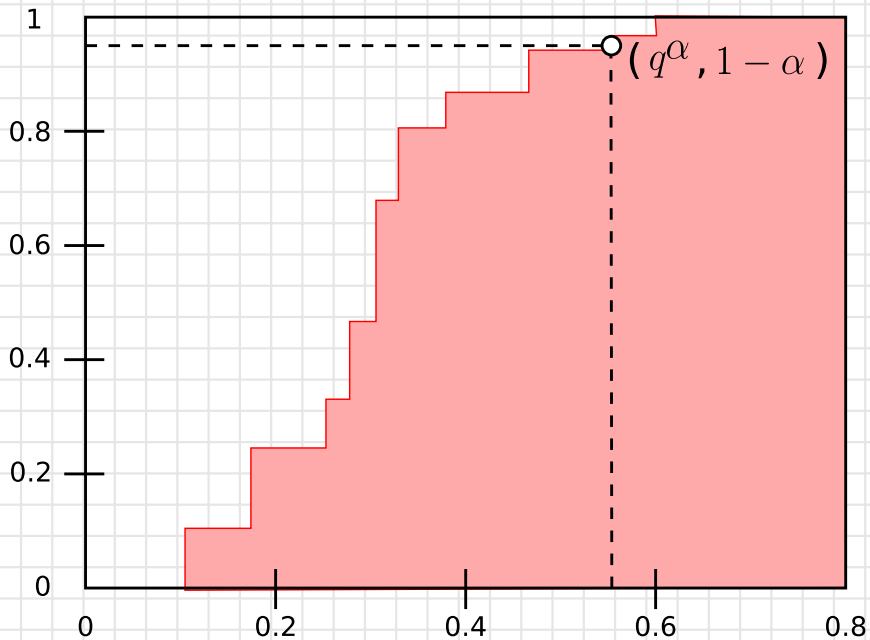

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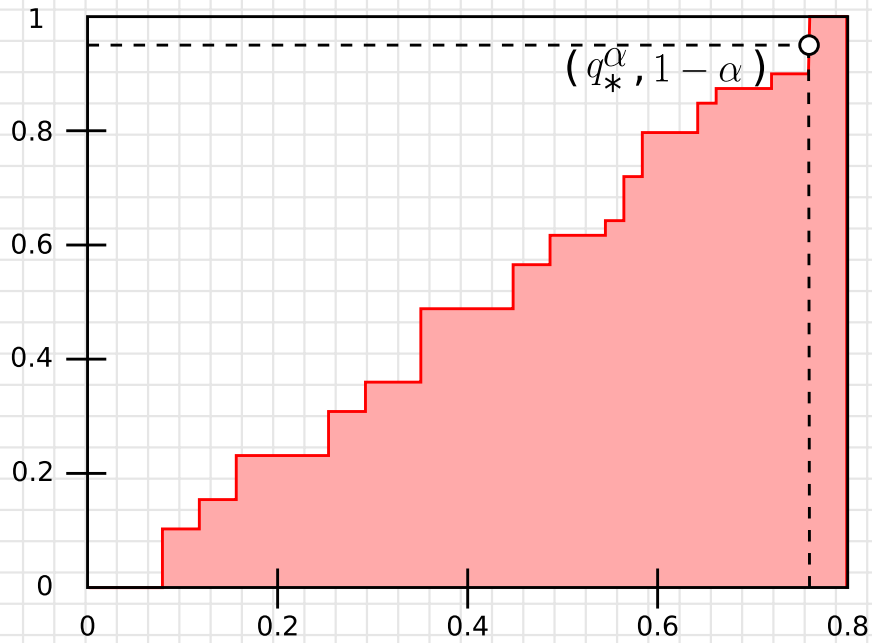
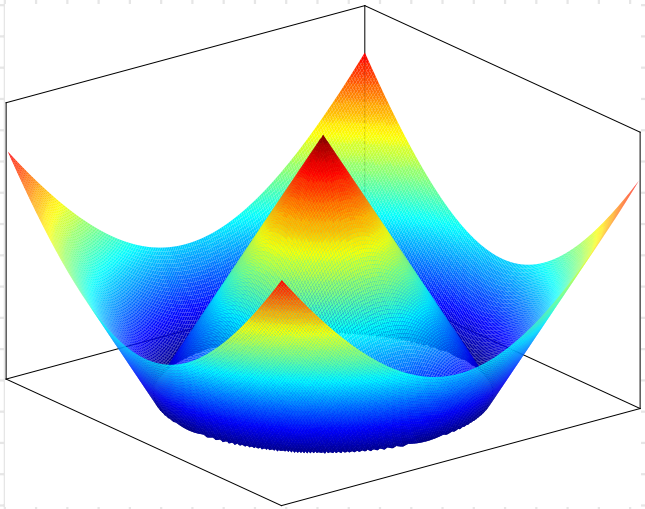


Distance to a Subset

$d_{\mathbb{M}}(a) = \inf_{x \in \mathbb{M}} ||x - a||$
$D = \mathrm{Dgm}_p^-(d_{\mathbb{M}})$



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P has continuous density p.
$\text{support}(P) = \mathbb{M}$.
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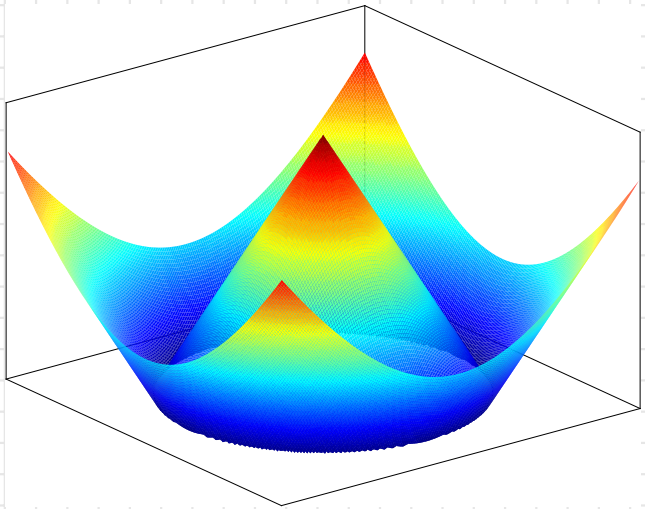

Subsampling

A Variant of the Bootstrap

Let $\mathcal{S}_b^i$ be a subsample of size $b < n$, for $i = 1, \ldots, B$.
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Confidence Sets from Subsampling [**F**LRWBS]

Assume that $p(x)$ is bounded away from zero.
Then, almost surely, for all large n,

$$\mathbb{P}\left(||d_{\mathbb{M}} - d_{\mathcal{S}_n}||_\infty > 2L_b^{-1}(\alpha)\right) \leq \alpha + O\left(\sqrt{1/n}\right)$$

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[RS-2002] On the uniform asymptotic validity of subsampling and the bootstrap. Annals of Statistics.

Putting It All Together

Subsampling Theorem

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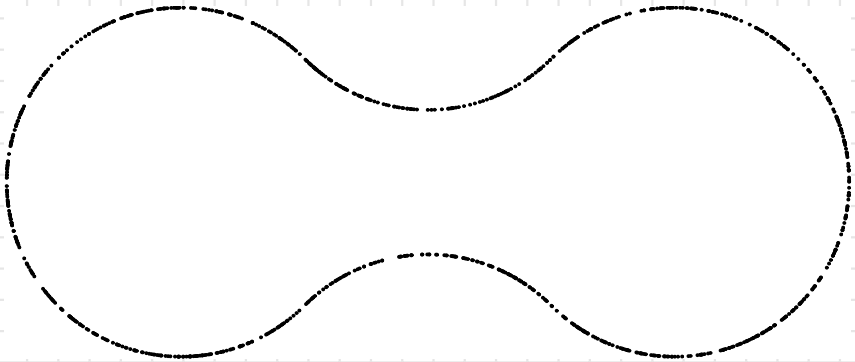
Asymptotic Confidence Sets for Persistence Diagrams

$$\lim_{n\to\infty} \mathbb{P}\left(W_\infty(D, \widehat{D}_n) \leq 2L_b^{-1}(\alpha)\right) \quad \geq \quad 1 - \alpha$$

Varying α



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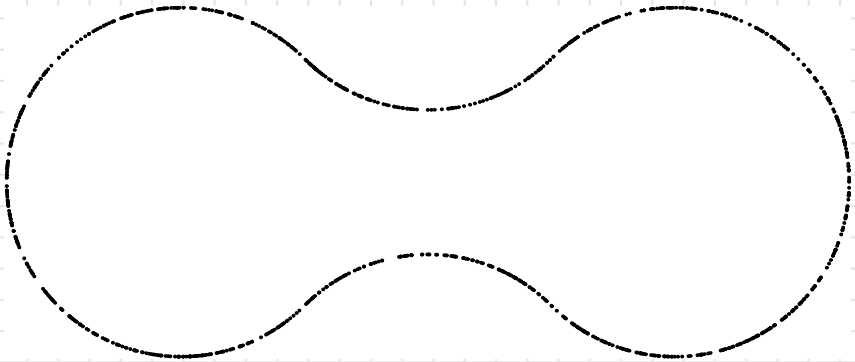
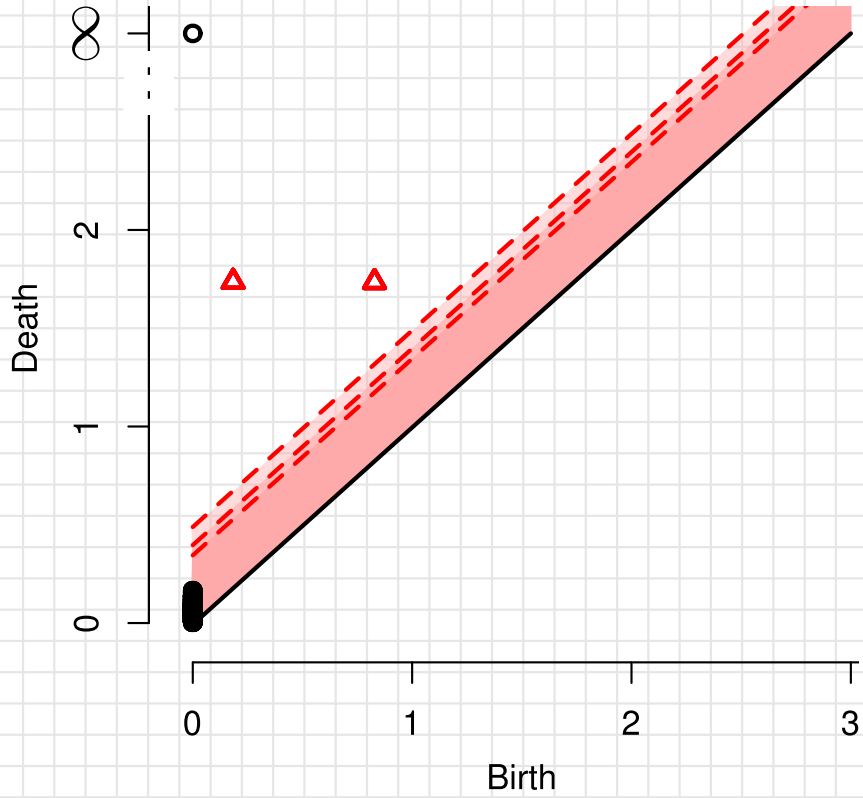




$$\alpha = 0.001, 0.05, 0.25$$

Two More Methods

$\mathcal{S}_n = \mathcal{S}_{1,n} \bigsqcup \mathcal{S}_{2,n}$.

Theorem (Concentration of Measure)

There exists $\hat{t}_{cm} = \hat{t}_{cm}(\alpha, d, n, \mathcal{S}_{1,n})$ *such that*

$$\mathbb{P}\left(W_\infty(D, \widehat{D}_n) > \hat{t}_{cm}\right) \leq \alpha + O\left(\left(\frac{\log n}{n}\right)^{1/d+2}\right).$$

Theorem (Method of Shells)

There exists $\hat{t}_s = \hat{t}_s(\alpha, d, n, K, \mathcal{S}_{1,n})$ *such that*

$$\mathbb{P}\left(W_\infty(D, \widehat{D}_n) > \hat{t}_s\right) \leq \alpha + O\left(\left(\frac{\log n}{n}\right)^{1/d+2}\right).$$

These Methods are Different

Concentration of Measure

$\hat{t}_{cm}$ is found by solving the following for t:

$$\frac{2^{d+1}}{t^d \hat{\rho}_{1,n}} \exp\left(-\frac{n t^d \hat{\rho}_{1,n}}{2}\right) = \alpha.$$

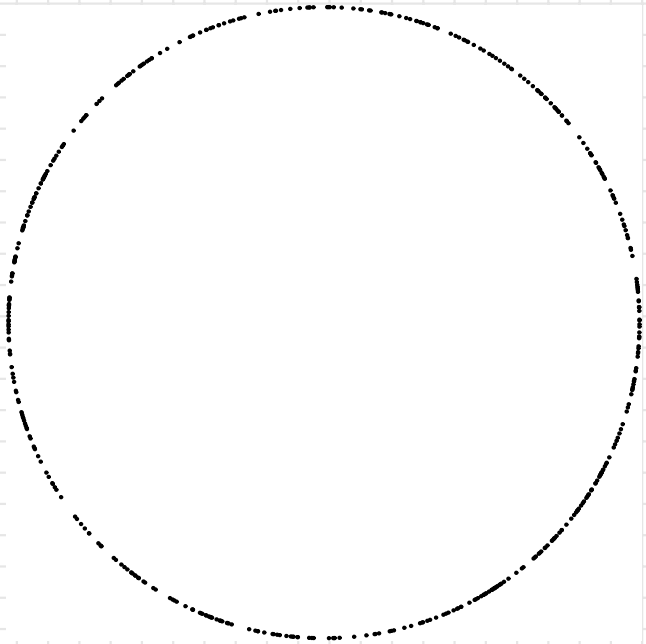
Shells

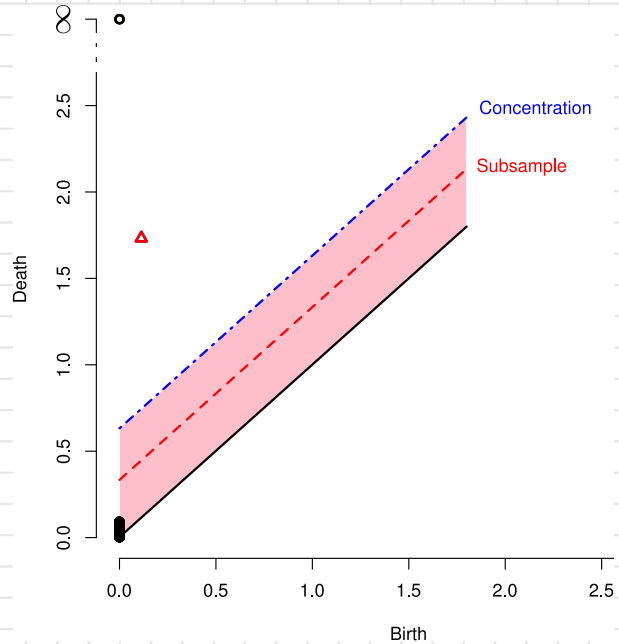
$\hat{t}_s$ is found by solving the following for t:

$$\frac{2^{d+1}}{t^d} \int_{\hat{\rho}_n}^{\infty} \frac{\hat{g}(v)}{v} \exp\left(-\frac{n v t^d \hat{\rho}_{1,n}}{2}\right) dv = \alpha.$$

Distance Function Examples

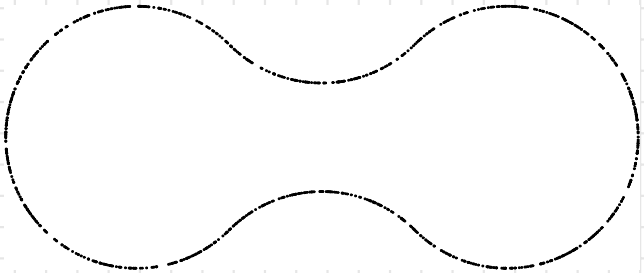
Uniform Distribution on Unit Circle

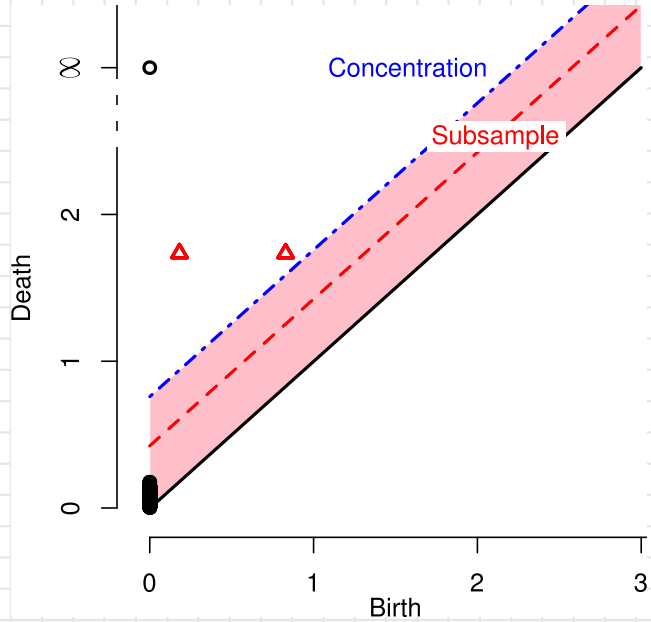




Distance Function Examples

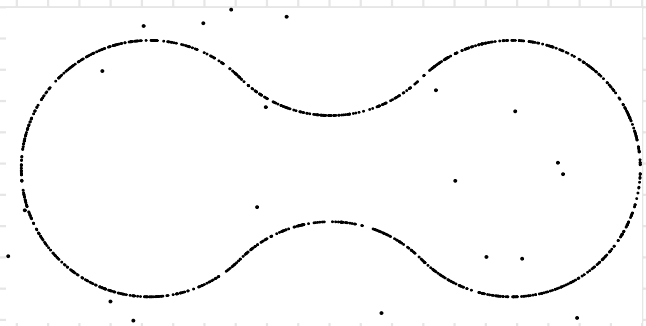
Uniform Distribution on Cassini Curve

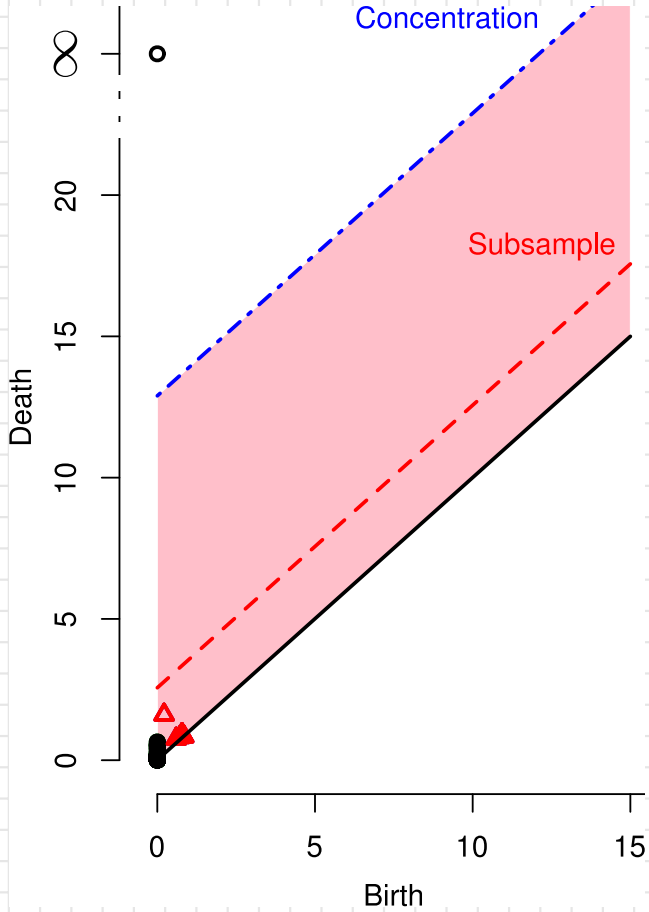




Distance Function Examples

Cassini Curve with Outliers





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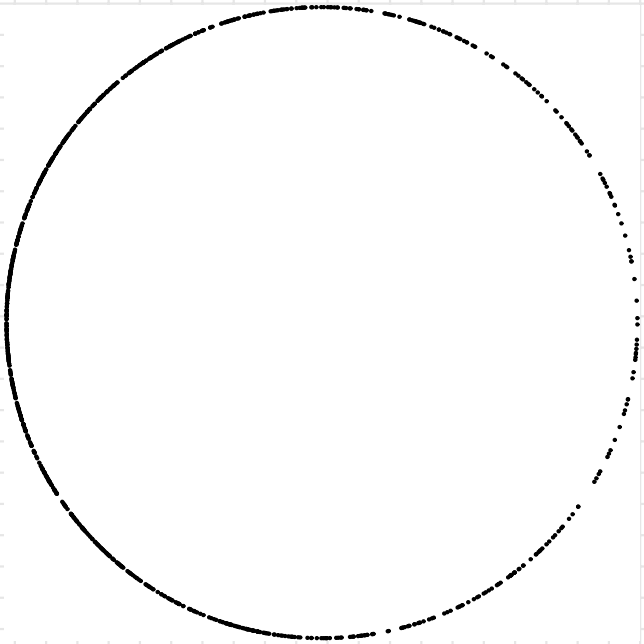
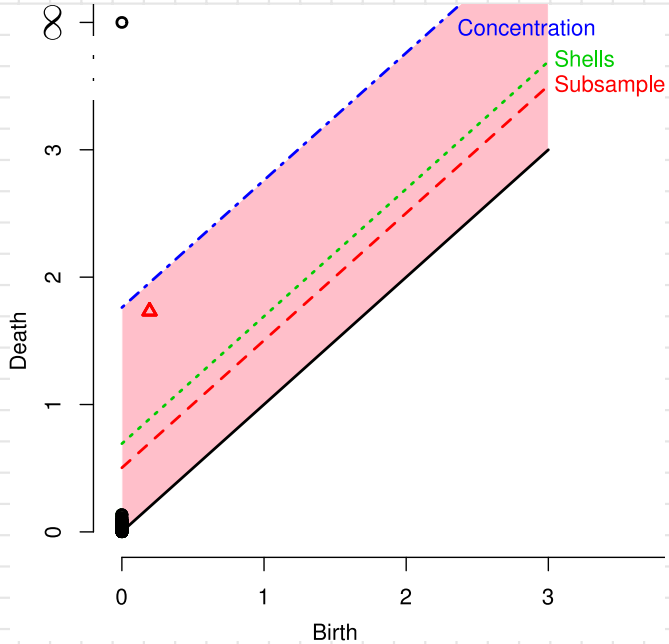




The Intersection of Statistics and Topology:

Stochastic Convergence of Persistence Landscapes and Silhouettes

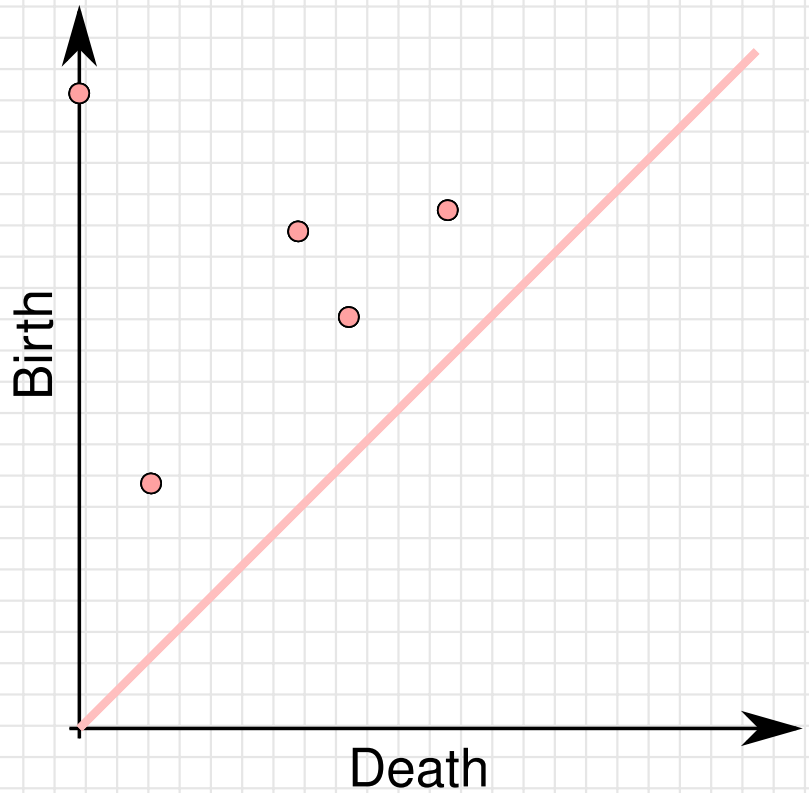
Brittany Terese Fasy

joint work with F. Chazal, F. Lecci, B. Michel
A. Rinaldo, L. Wasserman

3 June 2014

[C**F**LRW] Stochastic Convergence of Persistence Landscapes and Silhouettes. ArXiv. SoCG 2014 Proceedings, Kyoto. Journal version pending.

Persistence Landscapes



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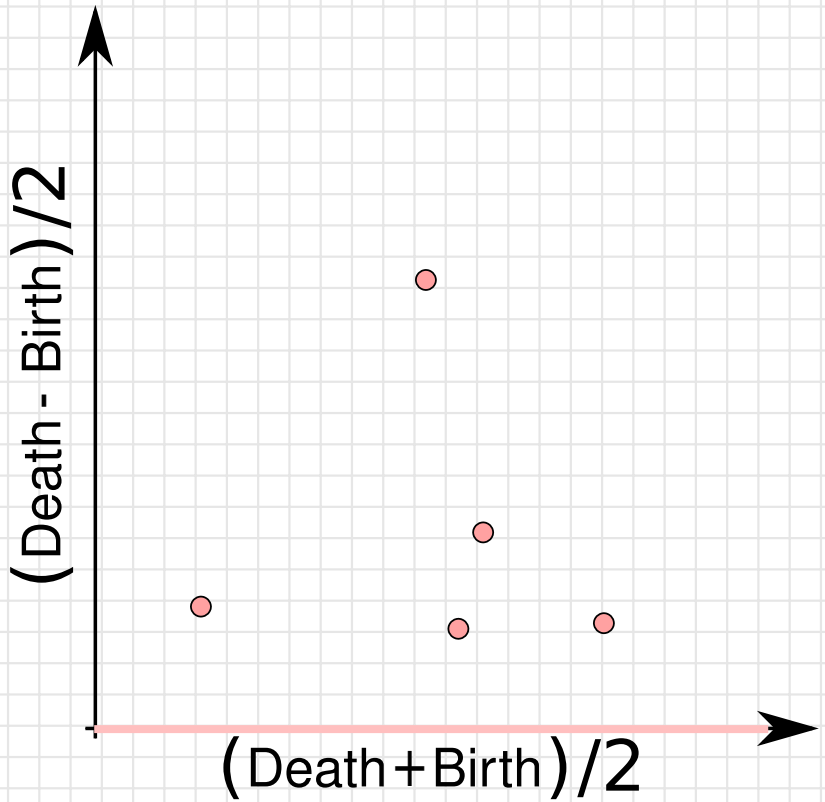

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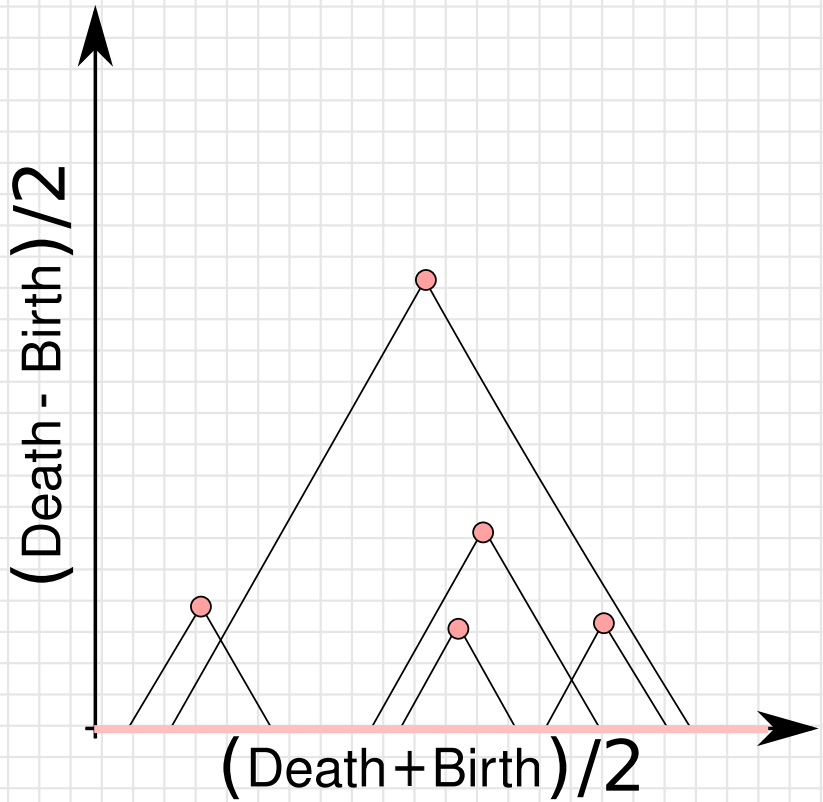

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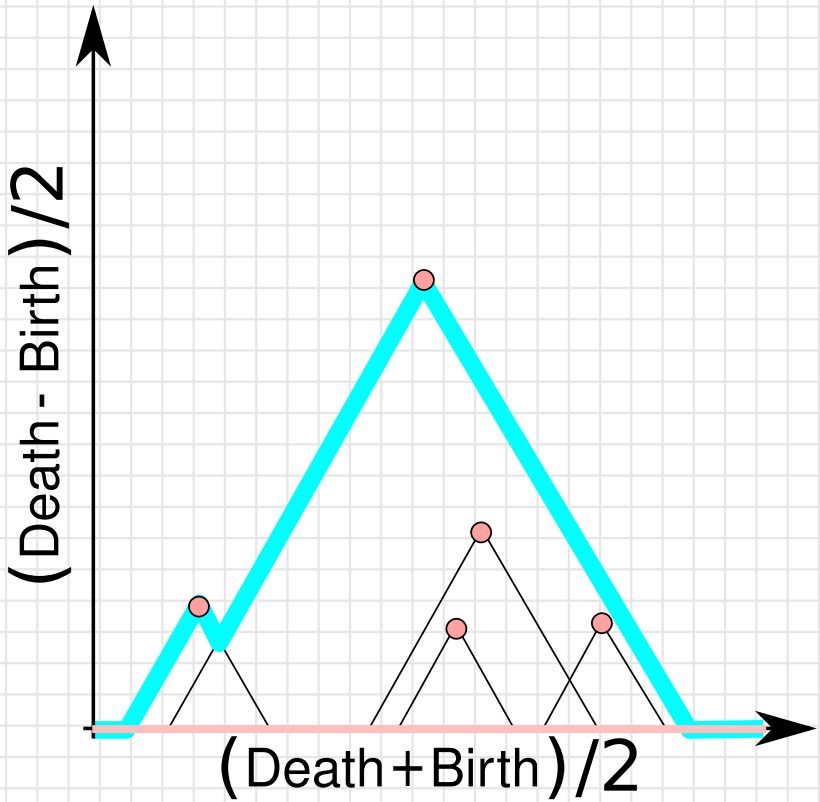
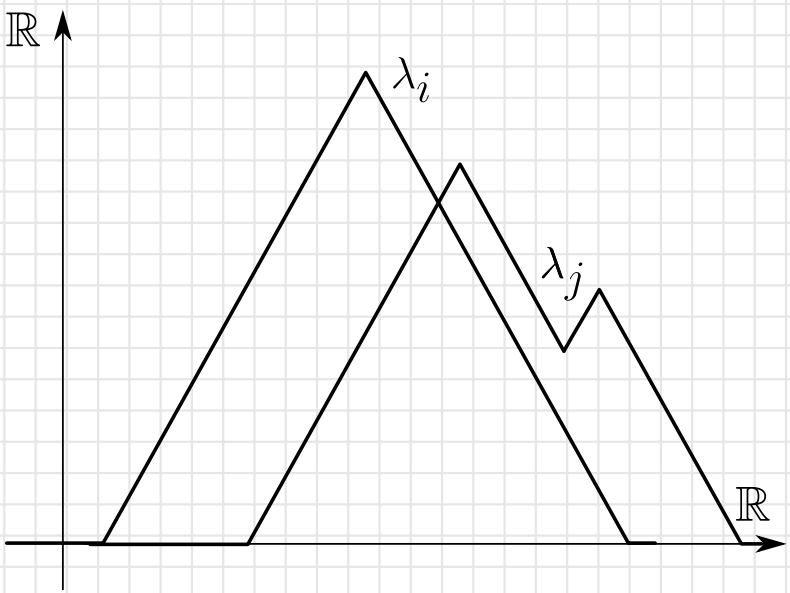


[B-2012] Statistical Topology Using Persistent Homology. ArXiv 1207.6437

Weak Convergence of Landscapes



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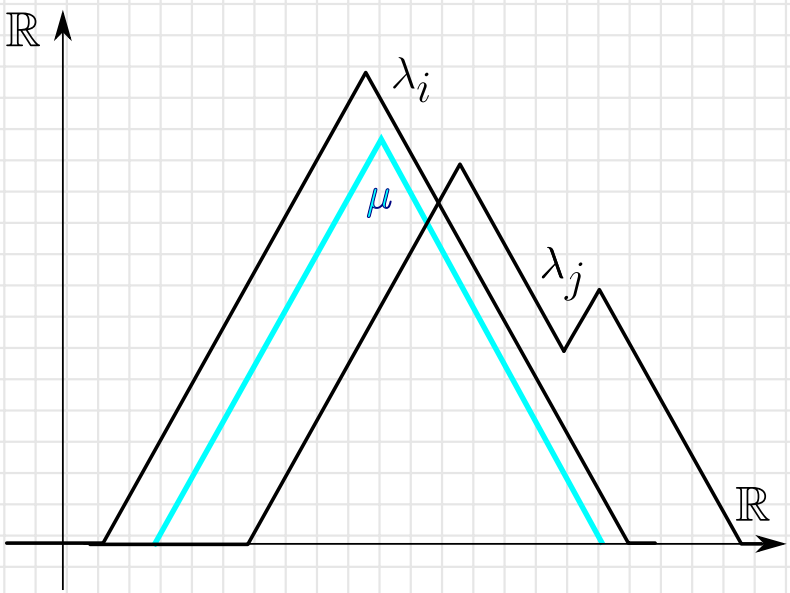

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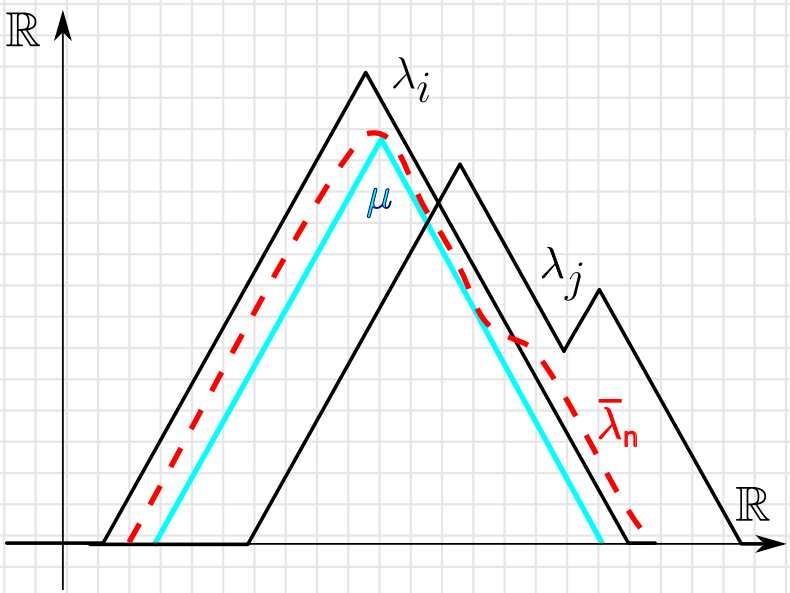

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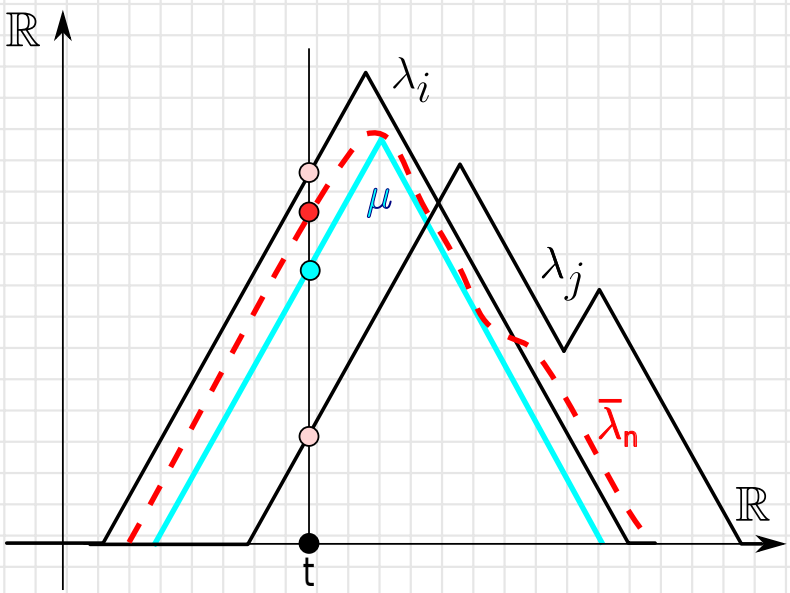

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$\bar{\lambda}_n$ converges pointwise to μ.

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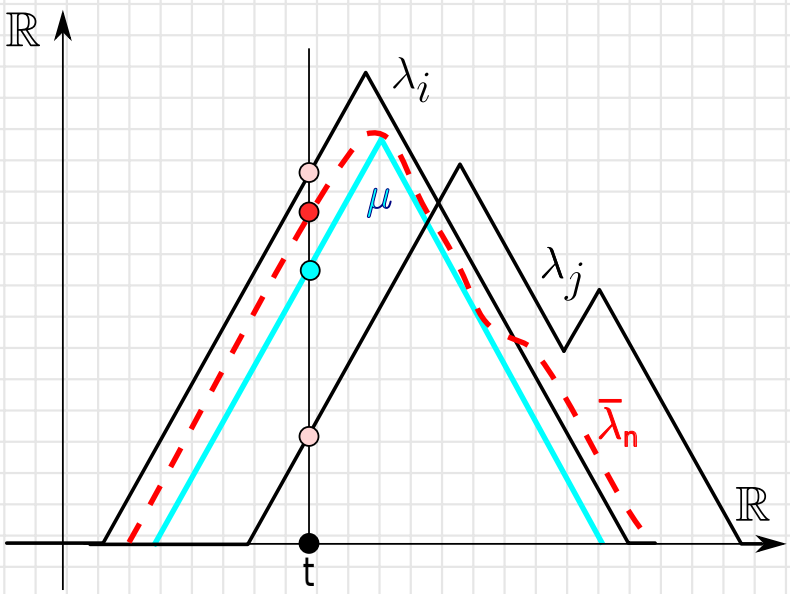

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Gaussian Process on $[0, T]$

For $t \in [0, T]$, we define $\mathbb{G}_n(f_t) = \mathbb{G}_n(t) := \frac{1}{\sqrt{n}}(\bar{\lambda}_n(t) - \mu(t))$.

Uniform Convergence

Weak Convergence

$\mathbb{G}_n$ converges weakly to the Brownian bridge $\mathbb{G}$ with covariance function

$$\kappa(f,g) = \int f(u)g(y)dP(u) - (\int f(u)dP(u))(\int g(u)dP(u)).$$

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$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}\Big(\sup_{t\in[t_*,t^*]} |\mathbb{G}_n(t)| \le z \Big) - \mathbb{P}(W \le z) \right| = O\Big(\frac{(\log n)^{\frac{7}{8}}}{n^{\frac{1}{8}}} \Big).$$

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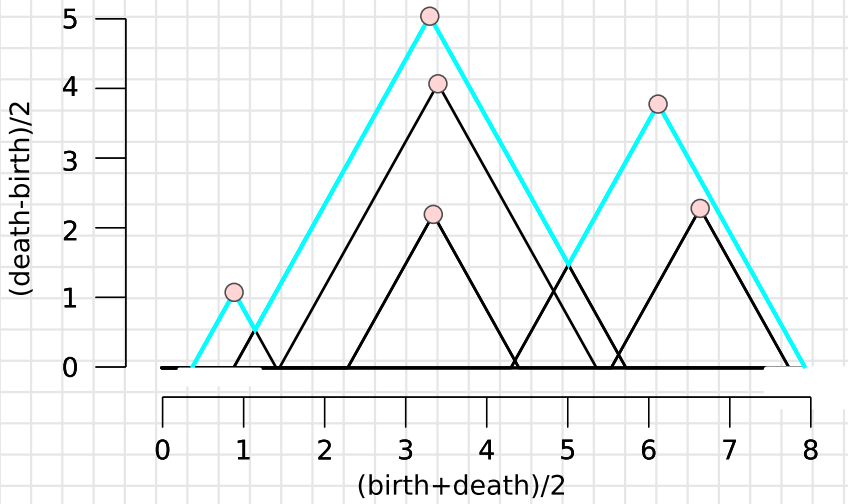
Uniform CLT

There exists a random variable $W \stackrel{d}{=} \sup_{t\in[t_*,t^*]} |\mathbb{G}(f_t)|$ such that

$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}\Big(\sup_{t\in[t_*,t^*]} |\mathbb{G}_n(t)| \leq z \Big) - \mathbb{P}(W \leq z) \right| = O\Big(\frac{(\log n)^{\frac{7}{8}}}{n^{\frac{1}{8}}} \Big).$$

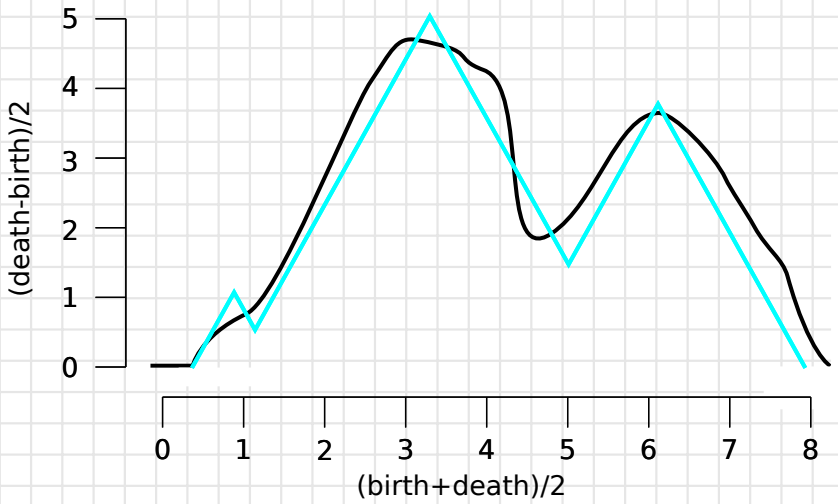
Proofs rely on [CCK-2013]: Anti-concentration and honest adaptive confidence bands. ArXiv 1303.7152.

Confidence Bands



$$\mathbb{E}||\lambda - \bar{\lambda}_n||_\infty \leq \overbrace{\mathbb{E}||\mu - \bar{\lambda}_n||_\infty}^{\text{variance}} + \underbrace{\mathbb{E}||\lambda - \mu||_\infty}_{\text{bais}}$$

Confidence Bands



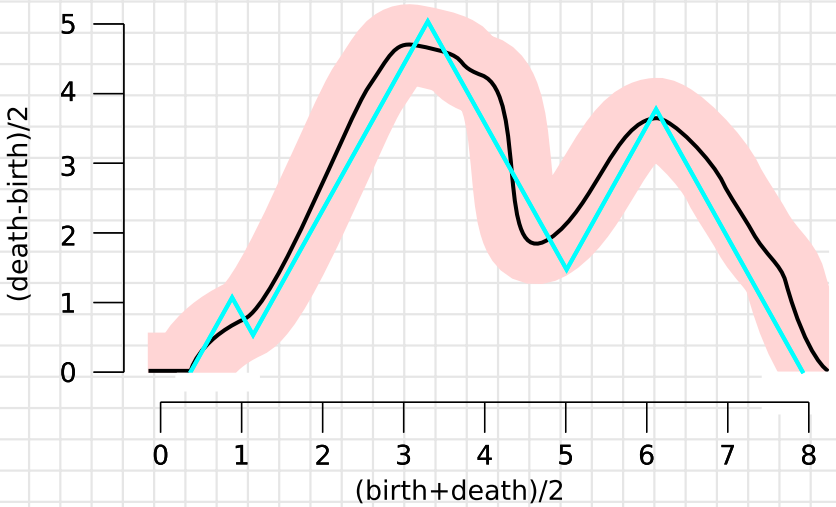
$$\mathbb{E}||\lambda - \bar{\lambda}_n||_\infty \leq \overbrace{\mathbb{E}||\mu - \bar{\lambda}_n||_\infty}^{\text{variance}}$$

Confidence Band

Find c such that $\ell_n = \bar{\lambda}_n - c$ and $u_n = \bar{\lambda}_n + c$ such that

$$\mathbb{P}(\ell_n(t) \leq \mu(t) \leq u_n(t) \text{ for all } t) \geq 1 - \alpha.$$

Asymptotic Confidence Bands



$$\mathbb{E}||\lambda - \bar{\lambda}_n||_\infty \leq \overbrace{\mathbb{E}||\mu - \bar{\lambda}_n||_\infty}^{\text{variance}}$$

Asymptotic Confidence Band

Find c such that $\ell_n = \bar{\lambda}_n - c$ and $u_n = \bar{\lambda}_n + c$ such that

$$\lim_{n\to\infty} \mathbb{P}(\ell_n(t) \leq \mu(t) \leq u_n(t) \text{ for all } t) \geq 1 - \alpha.$$

Confidence Bands for Landscapes

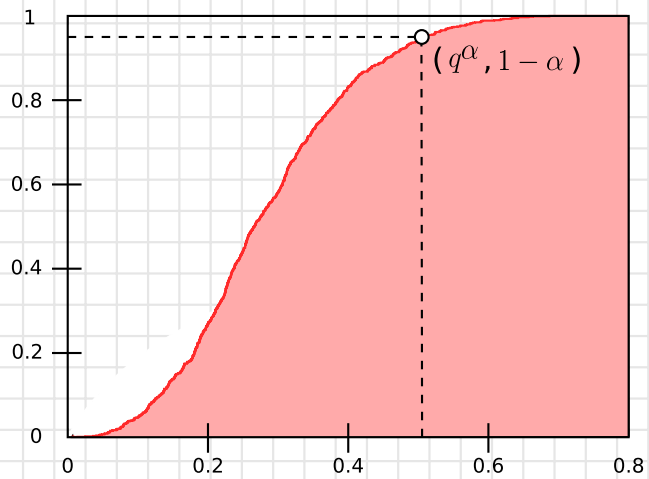
$\xi_1, \ldots, \xi_n \sim N(0,1)$

$$\widetilde{\mathbb{G}}_n(f_t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(\lambda_i(t) - \bar{\lambda}_n(t))$$

α-Quantile

$\widetilde{Z}_\alpha$ is the unique value such that

$$\mathbb{P}\left(\sup_t |\widetilde{\mathbb{G}}_n(f_t)| > \widetilde{Z}_\alpha \,\middle|\, \{\lambda_i\} \right) = \alpha$$



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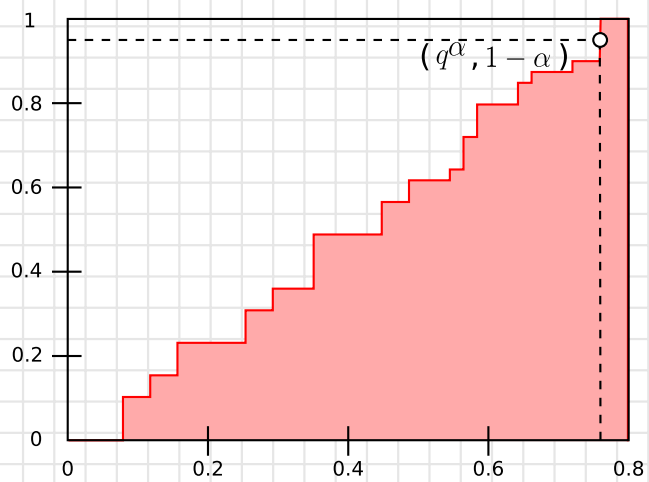
*Approx. $\widetilde{Z}_\alpha$ by MC simulation

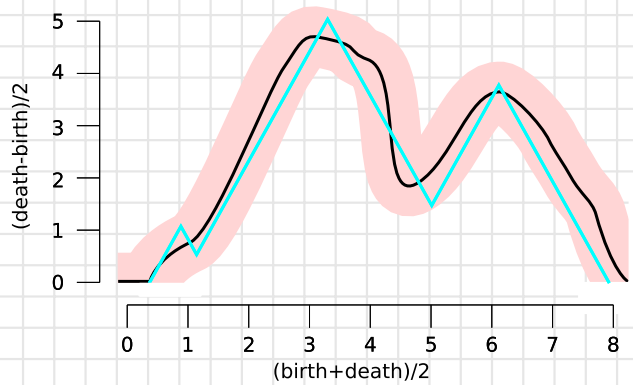


Confidence Bands for Landscapes

The Multiplier Bootstrap

$$\ell_n = \bar{\lambda}_n(t) - \frac{\tilde{Z}(\alpha)}{\sqrt{n}}$$

$$u_n = \bar{\lambda}_n(t) + \frac{\tilde{Z}(\alpha)}{\sqrt{n}}$$



Uniform Band

$$\mathbb{P}\left(\ell_n(t) \leq \mu(t) \leq u_n(t) \text{ for all } t\right) \geq 1 - \alpha - \left(\frac{(\log n)^{\frac{7}{8}}}{n^{\frac{1}{8}}}\right).$$

Variable Width Confidence Bands

$$\mathbb{H}_n(f_t) := \mathbb{G}_n(t)/\sigma(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\lambda_i(t) - \mu(t)}{\sigma(t)}$$

$$\widehat{\mathbb{H}}_n(f_t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \frac{\lambda_i(t) - \bar{\lambda}_n(t)}{\hat{\sigma}_n(t)}$$

$\widehat{Q}_\alpha$ is the unique value such that

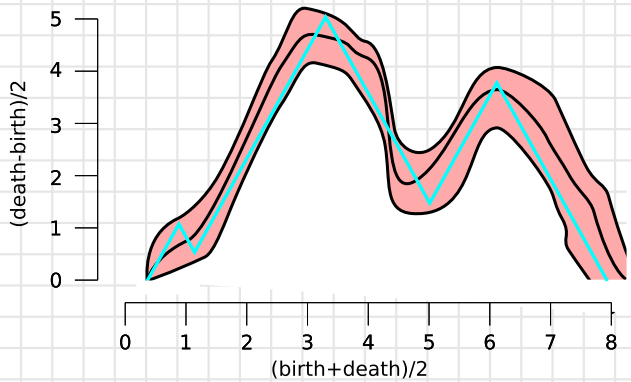
$$\mathbb{P}\left(\sup_t |\widetilde{\mathbb{H}}_n(f_t)| > \widetilde{Q}_\alpha \;\middle|\; \{\lambda_i\} \right) = \alpha$$

Confidence Bands for Landscapes

The Multiplier Bootstrap

$$\ell_n = \bar{\lambda}_n(t) - \frac{\hat{Q}(\alpha)\hat{\sigma}_n(t)}{\sqrt{n}}$$

$$u_n = \bar{\lambda}_n(t) + \frac{\hat{Q}(\alpha)\hat{\sigma}_n(t)}{\sqrt{n}}$$



Variable Band

$$\mathbb{P}\left(\ell_n(t) \leq \mu(t) \leq u_n(t) \text{ for all } t\right) \geq 1 - \alpha - \left(\frac{(\log n)^{\frac{7}{8}}}{n^{\frac{1}{8}}}\right).$$

The Intersection of Statistics and Topology:

... And More!

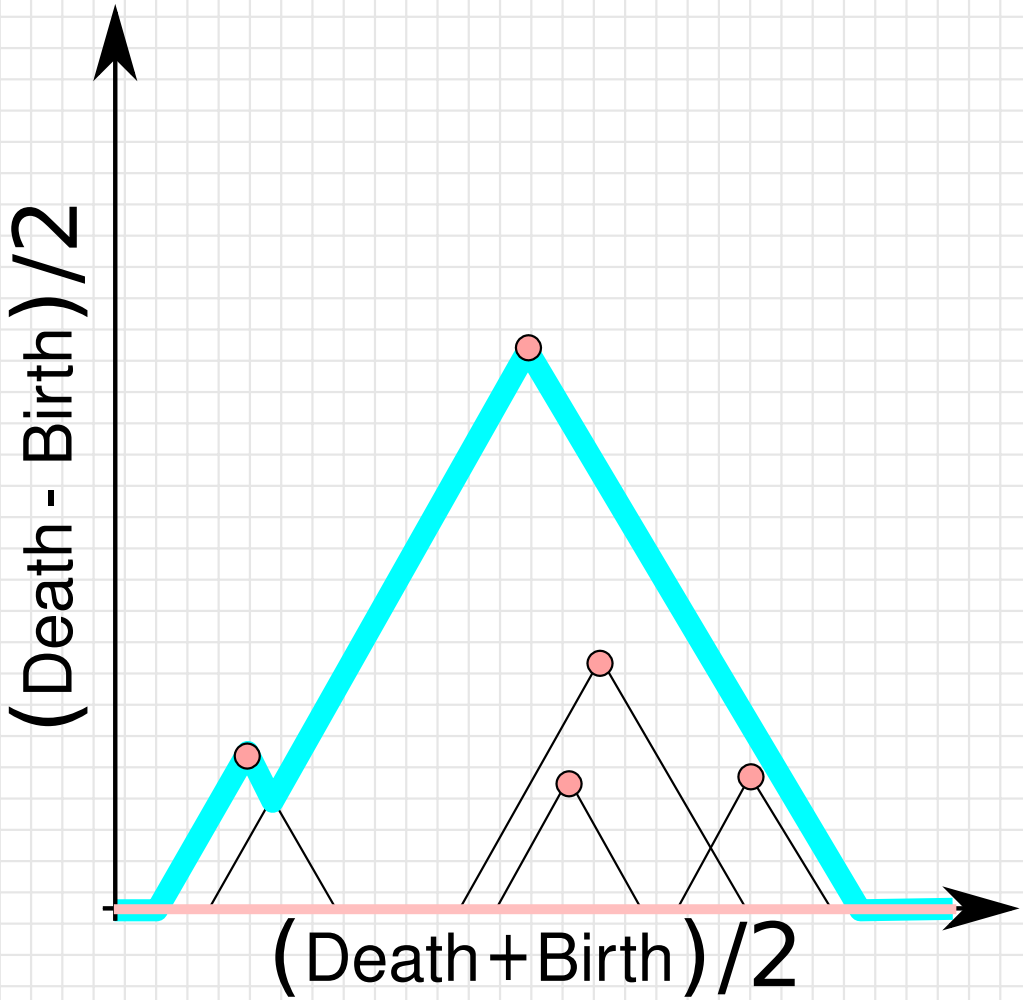
Brittany Terese Fasy

joint work with S. Balakrishnan, F. Lecci,
A. Rinaldo, A. Singh, L. Wasserman

3 June 2014

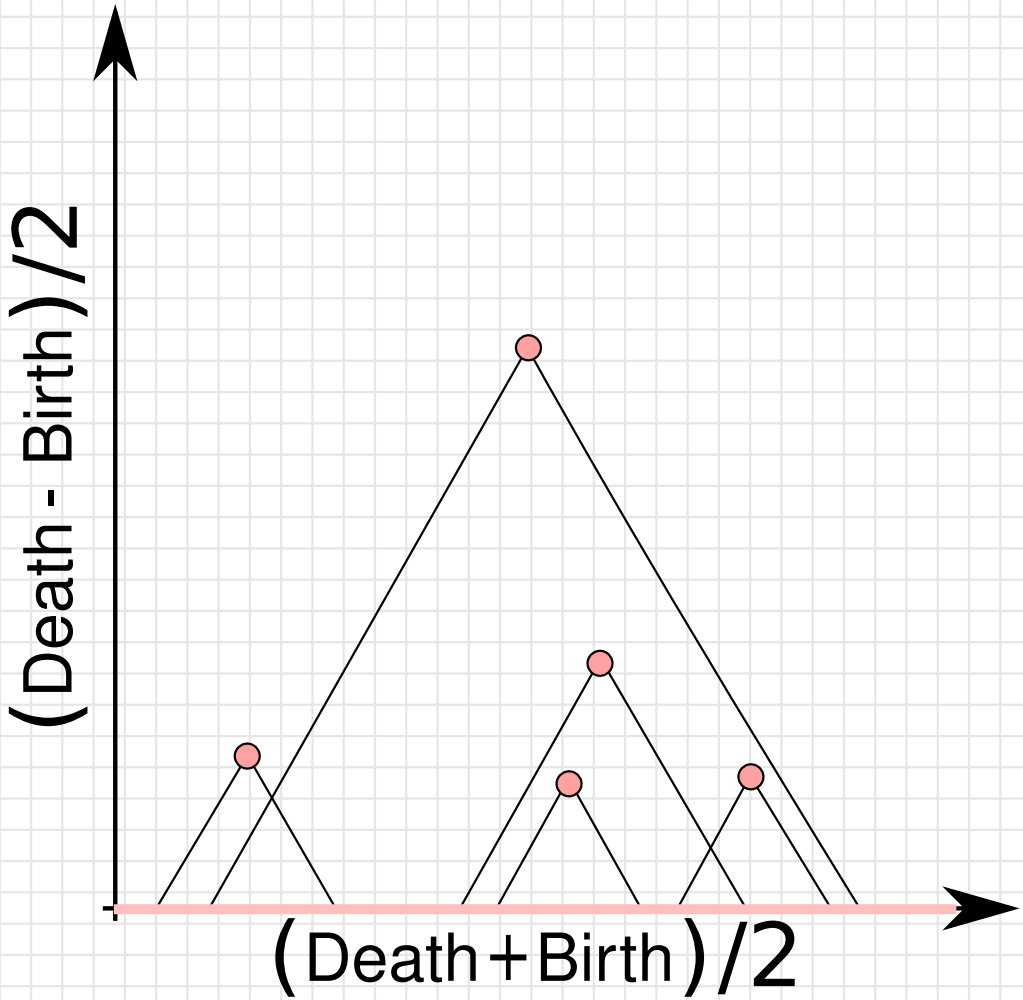
Persistence Silhouettes

Definitions



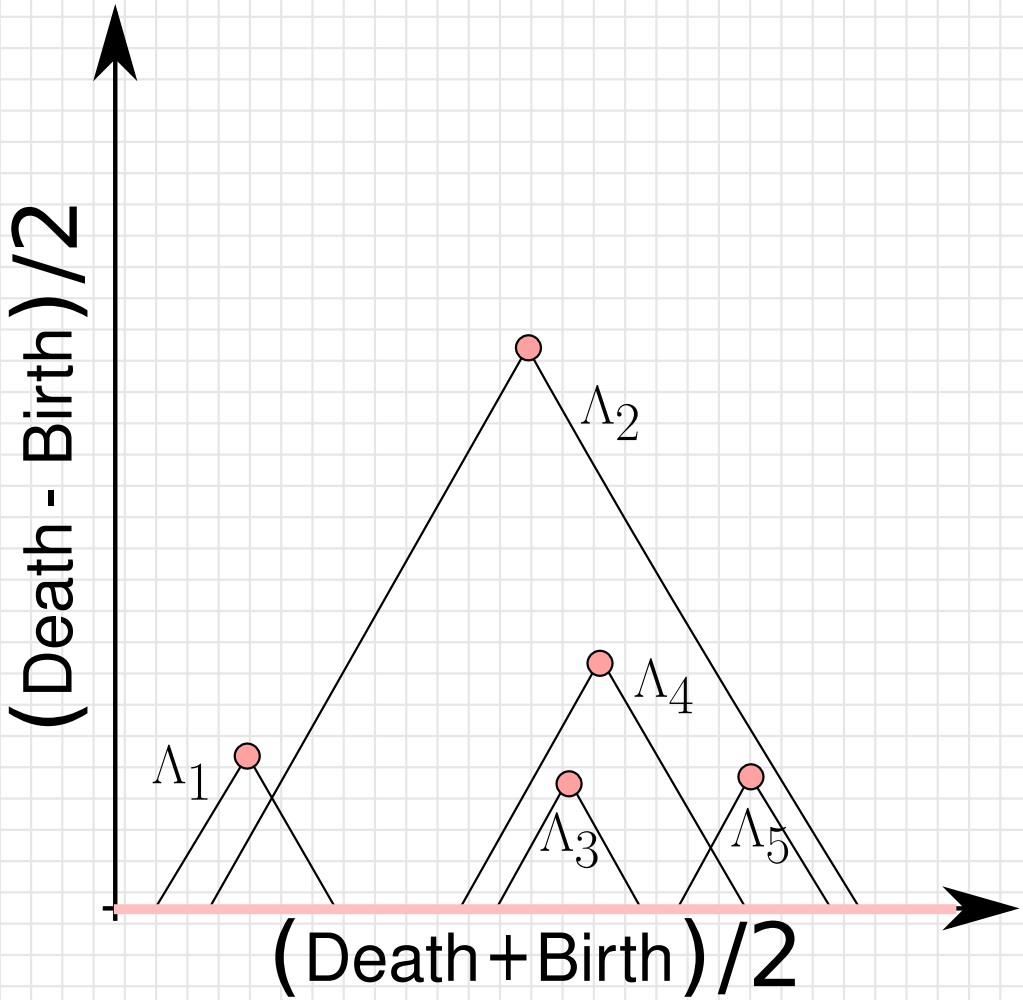
Persistence Silhouettes

Definitions



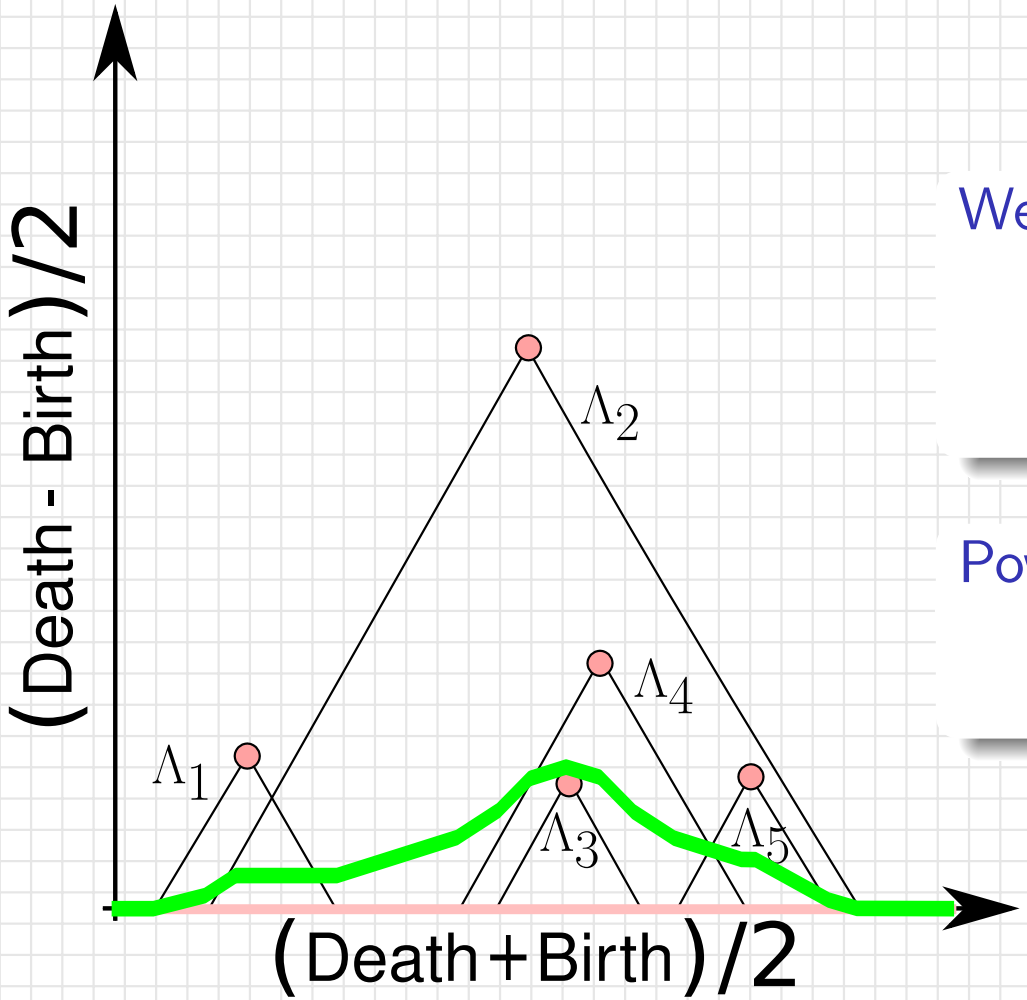
Persistence Silhouettes

Definitions



Persistence Silhouettes

Definitions



Weighted Silhouette

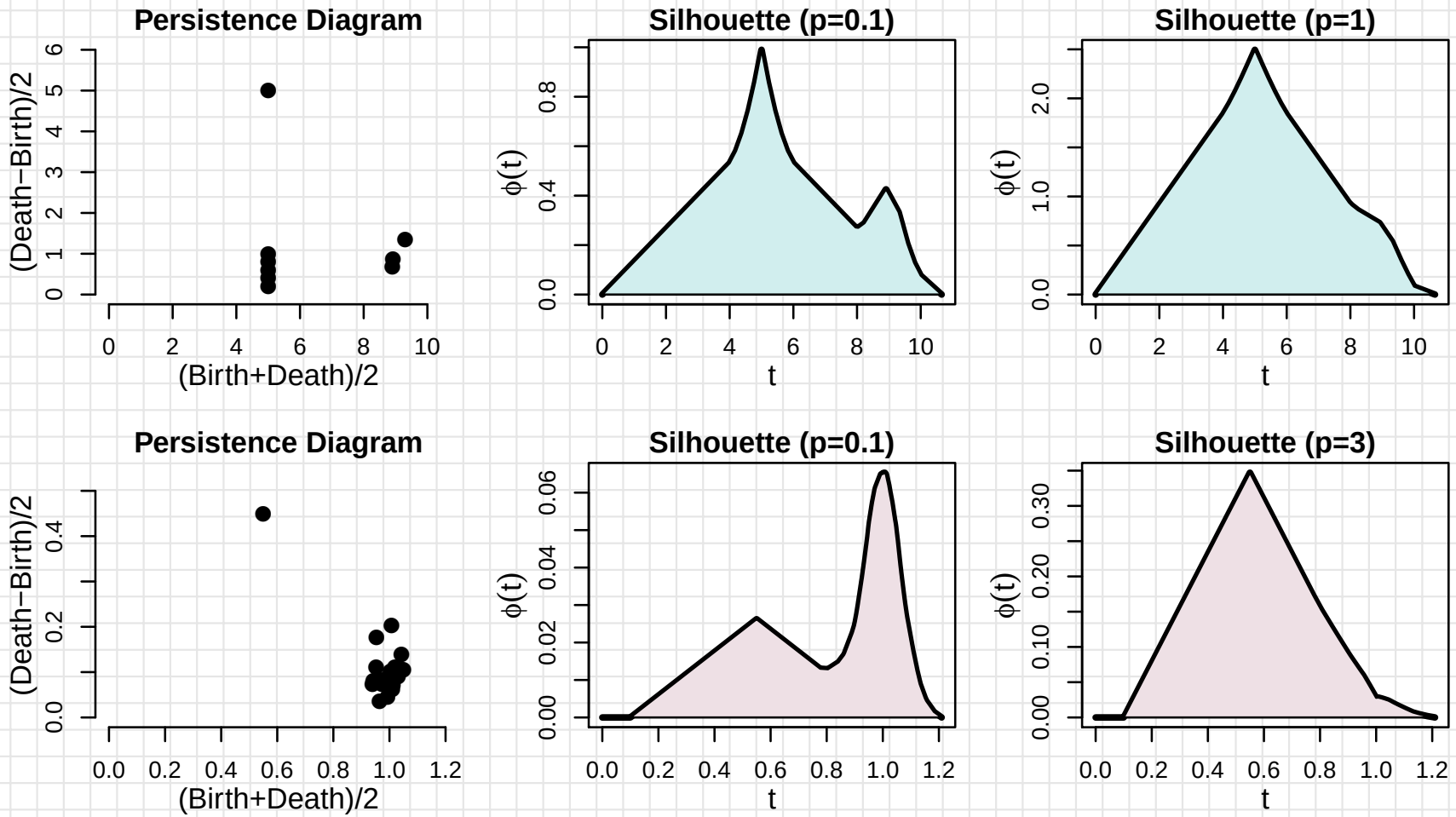
$$\phi(t) = \frac{\sum_{i=1}^{n} w_i \Lambda_j(t)}{\sum_{j=1}^{m} \sum w_j}$$

Power-Weighted Silhouette

$$w_i = |d_i - b_i|^p$$

Power-Weighted Silhouettes

Two Examples



Persistence Silhouettes

Results

Since ϕ is one-Lipschitz for non-negative weights w_j ...

Convergence of Empirical Process

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\phi_i(t) - \mathbb{E}[\phi(t)]\right)$$

converges weakly to a Brownian bridge, with known rate of convergence.

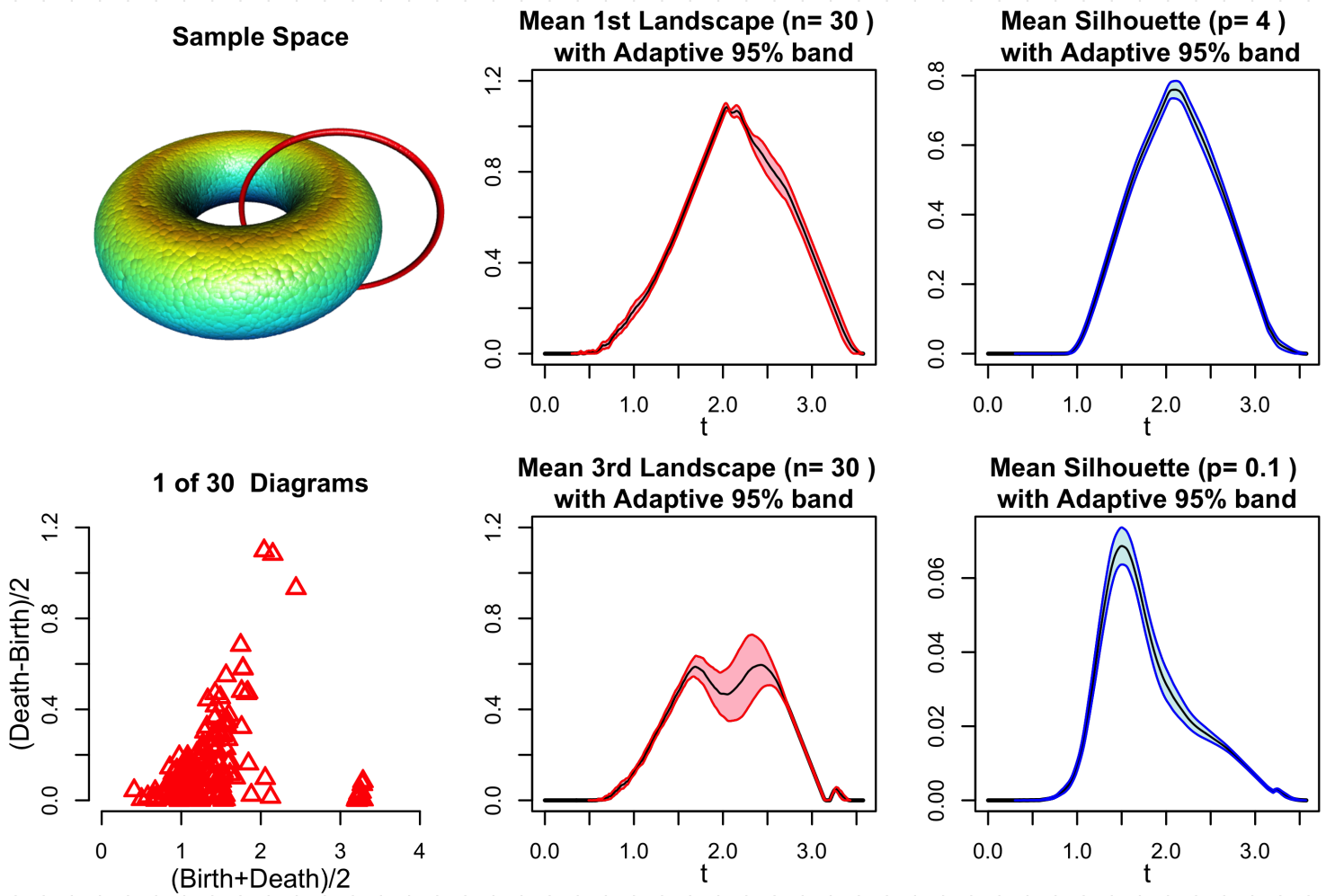
Confidence Bands

We can use the multiplier bootstrap to create a uniform (or a variable width) confidence band defined by ℓ_n^{sil} and u_n^{sil} such that

$$\lim_{n\to\infty}\mathbb{P}\left(\ell_n^{sil}(t) \le \mu(t) \le u_n^{sil}(t) \text{ for all } t\right) = 1-\alpha.$$

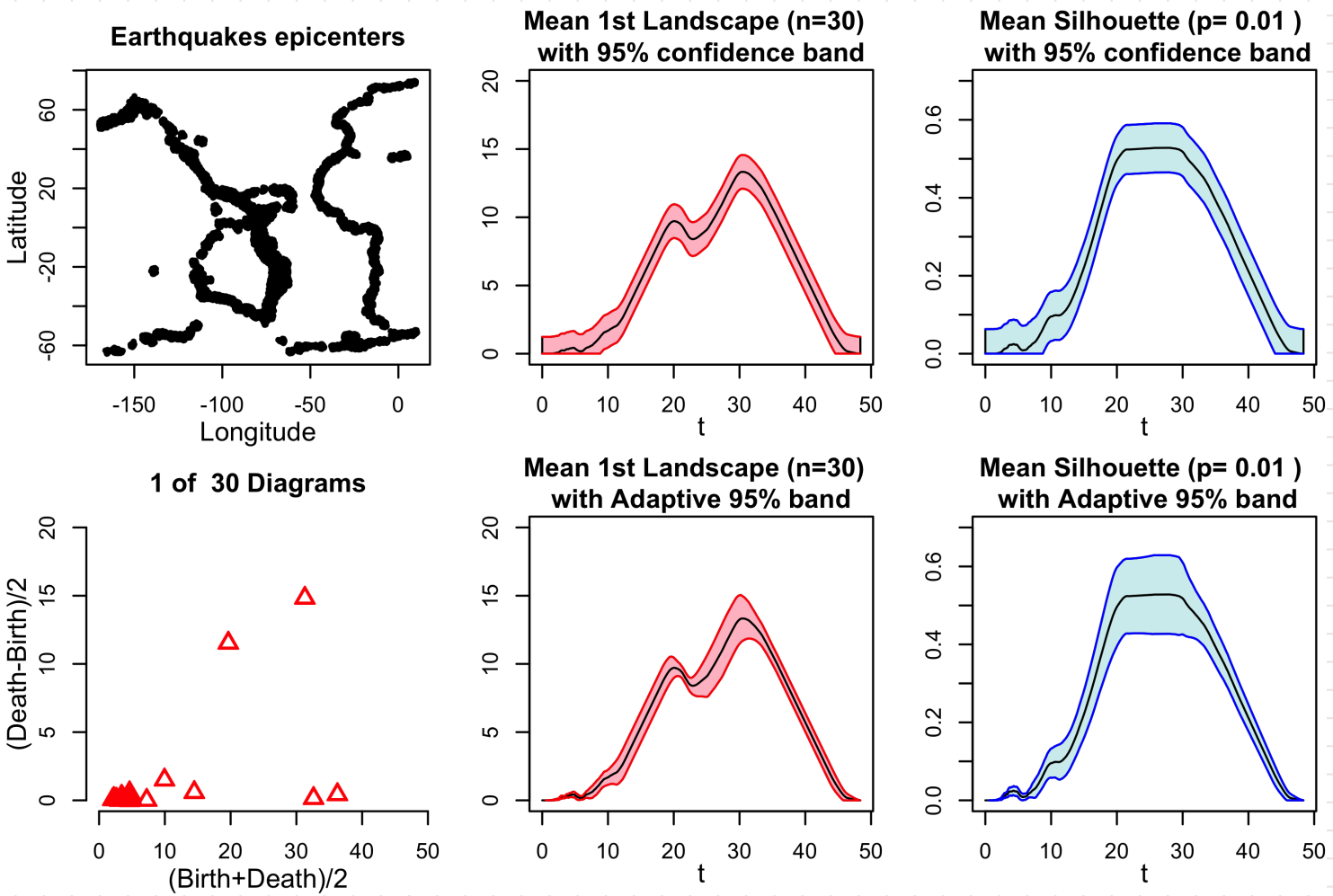
Example I

A Toy Example

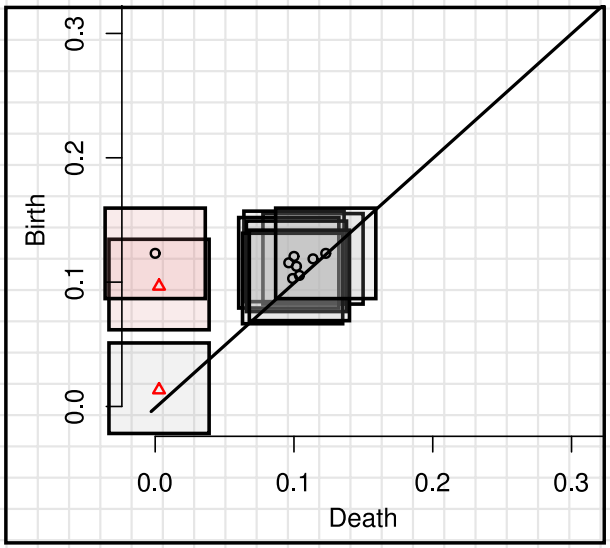


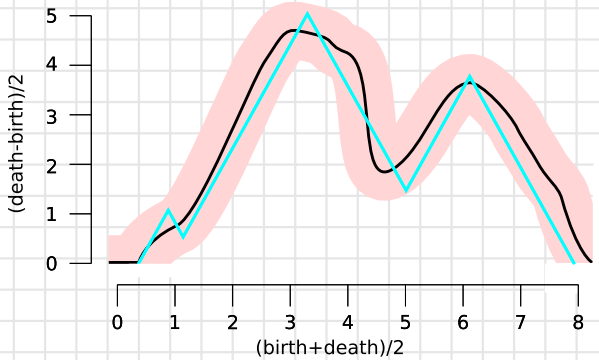
Example II

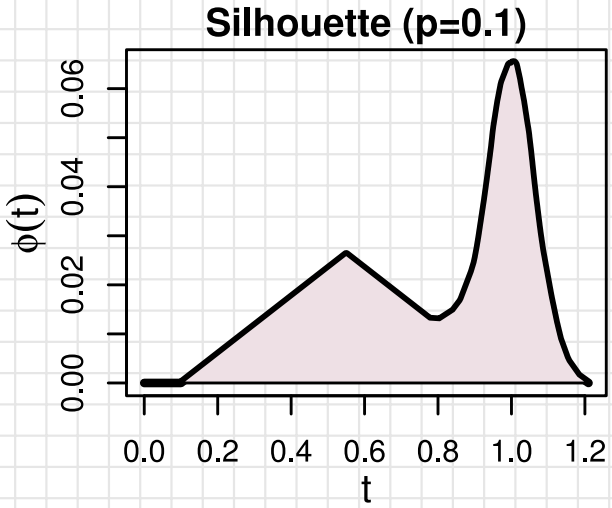
Earthquake Epicenters



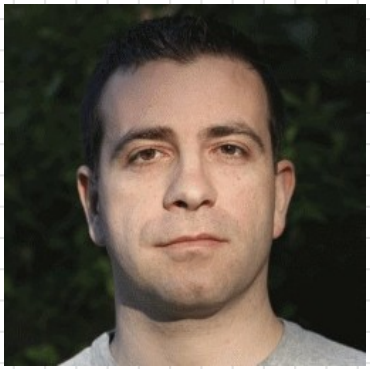
Summary


Birth
Death
0.0
0.1
0.2
0.3


(death-birth)/2
(birth+death)/2
0
1
2
3
4
5
6
7
8


Silhouette (p=0.1)
ϕ(t)
t
0.00
0.02
0.04
0.06
0.0
0.2
0.4
0.6
0.8
1.0
1.2

Collaborator Collage







Thank you!

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References

[CDGO] The Structure and Stability of Persistence Modules. ArXiv 1207.3674.

[C**F**LRSW] On the Bootstrap for Persistence Diagrams and Landscapes. Modeling and Analysis of Information Systems, **20**:6 (Dec. 2013), 96–105.