Sampling Weights, Model Misspecification and Informative Sampling: A Simulation Study

Marianne (Marnie) Bertolet*
Department of Statistics
Carnegie Mellon University

Abstract

Linear mixed-effects (LME) models analyze data that contain complex patterns of variability, specifically involving different nested layers. While LME models can match well the stratification and clustering of survey data, it is not clear how sampling weights should be incorporated into LME estimates. This report uses twelve simulation studies to compare two published methods of inserting sampling weights into LME estimates, Pfeffermann et al. (1998), denoted PSHGR, and Rabe-Hesketh and Skrondal (2006), denoted RHS. There are five main conclusions based on these simulations. 1) The PSHGR and RHS point estimates are very similar, with differences due to numerical instabilities in the estimation procedures. 2) Confidence intervals based on the sandwich estimator and the design based estimator of the variances provide similar coverage when there is no model misspecification. However, when there is model misspecification, the design-based variance estimator has unexpectedly large coverage, implying that the variance estimates are too large. 3) When there is model misspecification that does not induce informative sampling, weighted estimates do not reduce bias of the estimators. 4) When there is informative sampling, the weighted estimators do reduce the bias of the point estimates, though they do not eliminate it. 5) The unweighted estimate has the smallest variance. When there is informative sampling, the unweighted estimates are biased. The weighted unscaled estimate corrects the bias in the fixed effects, but produces more bias in the random effects. The scaled 1 weightings remove the bias in the fixed effects, and overcorrect for the weighted unscaled bias in the random effects. The scaled 2 weightings remove the bias in the fixed effects and are in between the weighted unscaled and weighted scaled 1 bias in the random effects.

Keywords: Linear mixed-effects models, survey sampling, weighting bias, sampling bias, sandwich estimator, design-based estimators.

*This work is based on my thesis, completed at Carneige Mellon University. Current affiliation is with the Epidemiology Data Center, Department of Epidemiology, Graduate School of Public Health, University of Pittsburgh, bertoletm@edc.pitt.edu

1 Introduction

Linear mixed-effects (LME) models analyze data that contain complex patterns of variability, specifically involving different nested layers. While LME models can match well the stratification and clustering of survey data, the debate continues whether or not sampling weights should be used and, if used, how they should be incorporated into LME estimates. This report analyzes two published methods of inserting sampling weights into LME estimates, Pfeffermann et al. (1998), denoted PSHGR, and Rabe-Hesketh and Skrondal (2006), denoted RHS. The specific goals are: 1) To compare the results from the different methods of inserting weights into LME models, 2) To compare the sandwich estimator of the variance of the point estimates to the design-based estimator, 3) To compare the results that use different scalings of the weights, 4) To investigate the assertion that adding sampling weights can compensate for informative sampling in LME models and 5) To investigate the assertion that adding sampling weights can compensate for model misspecification in LME models. Results of the simulation studies are presented for side-by-side comparisons of parameter estimates under different simulated conditions.

Section 3 summarizes the previous simulation studies, including their designs and results. Section 4 provides a description of the format of the new simulation results presented in this dissertation. Section 5 describes and presents results from the 12 new simulations. Section 6 compares the simulations with respect to a mean squared error metric. Section 7 summarizes the results from the 12 new simulations and explain how these new results verify and expand the previous simulation results. Finally, Section 8 contains a technical appendix.

The main contribution of this report are the 12 simulation sets and the conclusions from them. There are five main conclusions based on these simulations. 1) The PSHGR and RHS point estimates are very similar. The differences in the point estimates are due to numerical instabilities in the estimation procedures. 2) Confidence intervals based on the sandwich estimator and the design based estimator of the variances provide similar coverage when there is no model misspecification. However, when there is model misspecification, the design-based variance estimator has unexpectedly large coverage, implying that the variance estimates are too large. 3) When there is model misspecification that does not induce informative sampling, weighted estimates do not reduce bias of the estimators. 4) When there is informative sampling, the weighted estimators do reduce the bias of the point estimates, though they do not eliminate it. 5) The unweighted estimate has the smallest variance. When there is informative sampling, the unweighted estimates are biased. The weighted unscaled estimate corrects the bias in the fixed effects, but produces more bias in the random effects. The scaled 1 weightings remove the bias in the fixed effects, and overcorrect for the weighted unscaled bias in the random effects. The scaled 2 weightings remove the bias in the fixed effects and are in between the weighted unscaled and weighted scaled 1 bias in the random effects.

2 Simulation Goals and Summary of Results

As mentioned above, there are five specific goals for this report. In this section I describe each of them and provide a summary of the results from the simulations.

The first goal is to compare the results from the different methods of inserting weights into LME models. There are three published methods on inserting weights into LME models, Rabe-Hesketh and Skrondal (2006), denoted RHS, Korn and Graubard (2003), denoted KG, and Pfeffermann et al. (1998), denoted PSHGR. These methods use pseudo-maximum likelihood methods and differ in the location during the maximum likelihood estimation where the census quantities are estimated with weighted sample quantities. Asparouhov (2006), denoted ASP, published the same procedure as RHS at the same time. I focus on the RHS method, as opposed to the ASP method, as the software to implement RHS was available to me whereas the software to implement ASP was not available to me. The simulations in this report compare the RHS and PSGHR methods, as the KG method requires univariate, bivariate, trivariate and quadvariate conditional weights

($w_{i|k}, w_{ij|k}, w_{ijs|k}$ and $w_{lmst|k}$) that are generally not available. These simulations found that the RHS and PSHGR methods provide remarkably similar results. The differentiation between the methods is that the software that implements RHS (the `gllamm()` function in Stata) is not always numerically stable. This is due to the numerical quadrature implemented for the RHS method. For more details on the numerical instabilities, see Section 8.3.

The second goal is to compare the sandwich estimator (used by RHS) and a design-based estimator (used by PSHGR) when obtaining the variances of the point estimates. It appears that when there is no model misspecification, that the confidence intervals based on the sandwich estimator have similar coverage levels as the confidence intervals based on the design-based estimates. However, when there is model misspecification, the design-based confidence intervals have coverage that is unexpectedly large, implying that the variance estimates are too large.

The third and fourth goals of this report, described below, relate to the controversy of including sampling weights in model-based analyses. This controversy has been extensively debated, including but not limited to Fienberg (1989), Hoem (1989), Kalton (1989), Mislevy and Sheehan (1989), Thomas and Cyr (2002), Patterson et al. (2002) and Little (2004).

The third goal of this report is to investigate the assertion that adding sampling weights can compensate for model misspecification in LME models. The simulations in this chapter indicate that the weights can help for model misspecification only when the model misspecification induces informative sampling. Bias related to a misspecified model that does not relate to the sampling design are unaffected by the sampling weights.

The fourth goal of this report is to investigate the assertion that adding sampling weights can compensate for informative sampling in LME models. The simulations in this chapter support those conclusions. The inverse sampling weights can help compensate for bias induced by informative sampling, though they do not eliminate the bias.

The last goal of this report is to investigate the different scalings of the weights, denoted as unweighted, weighted unscaled, weighted scaled1 and weighted scaled 2 PSHGR and RHS. The weighted LME estimates are consistent if the number of clusters increases as the population size increases. If the conditional weights ($w_{i|k}$, the inverse probability that individual i is sampled provided cluster k is in the sample) are multiplied by a cluster level constant, then the consistency argument remains unchanged. This allows us to consider scalings of the weights to reduce the bias in the variance components. The simulations in this chapter compare the different scalings when the data are not balanced and when the models are more complicated than random intercept models. These simulations found that the unweighted estimate has the smallest variance. When there is informative sampling, the unweighted estimates are biased. The weighted unscaled estimate corrects the bias in the fixed effects, but produces more bias in the random effects. The scaled 1 weightings remove the bias in the fixed effects, and overcorrect for the weighted unscaled bias in the random effects. The scaled 2 weightings remove the bias in the fixed effects and are in between the weighted unscaled and weighted scaled 1 bias in the random effects.

This report also contains a number of appendices collected together in Section 8 that provide additional detail about the simulation methods and results. In particular, Section 8.6 summarizes the computer code written to run the simulations and provides web-links to the code for the interested reader.

3 Previous LME Simulation Results

3.1 Overview

Table 1 contains a summary of the previous simulation designs performed by the authors of the methods described in this thesis. The method by RHS was also published concurrently by ASP, whose simulation results are included in Table 1. This order of the presentation represents the order in which the weights are added; RHS (and ASP) insert the weights before the derivative is taken, KG insert the weights immediately after the derivative is taken, and PSHGR insert the weights in the process of solving for the parameter values.

In evaluating the previous studies with respect to the goals of this chapter, note that none of the authors compared their method to the other methods presented in this thesis, so there are no previous direct comparisons. All the authors' estimating models matched their generating models, so there was no model misspecification in previous simulations. Below, I summarize the authors' studies based upon the third and fourth goals listed above; to investigate the effect of weights on informative sampling and to compare the different scalings of the weights. In addition to my goals listed above, many of the authors were interested in the effect of sample sizes on the estimates and these are also listed in Table 1. Finally, I will also note the authors' methods of computing variances of their point estimates.

3.2 RHS Simulation Summary

Rabe-Hesketh and Skrondal (2006), denoted RHS, performed simulations with a logistic random intercept model, one cluster level covariate, x_{1k}, and one individual level covariate, x_{2ik},

$$\log\left(\frac{P(Y_{ik}=1)}{1-P(Y_{ik}=1)}\right) = 1 + x_{1k} + x_{2ik} + U_{0k}$$

Their finite population contains 500 clusters, each with the same number of elements per cluster (either 5, 10, 20, 50 or 100). They oversample clusters whose absolute value of the random effect (U_{0k}) was less than one and oversample individuals whose random error (ϵ_{ik}) is less than zero. They sample approximately 300 clusters and approximately half of the elements in the sampled cluster. The RHS results are summarized in Table 2. For this table, an estimate was labeled biased if the confidence interval (mean over 100 iterations $\pm$ 2 times standard deviation of the 100 iterations divided by 10) did not contain the true

		RHS	ASP	KG	PSHGR
	Simulation Comparions	None	None	None	None
Generated (and Estimated) Model	Random Intercept Model: $Y_{ik} = \beta_0 + U_{0k} + \epsilon_{ik}$		✓	✓	✓
	Logistic Random Intercept Model: $\text{logit}(Y_{ik}) = \beta_0 + \beta_1 x_{1ik} + \beta_2 x_{2k} + U_{0k}$	✓	✓		
	Two level model	✓	✓	✓	✓
	Multiple-level model		✓		
Sampling Scheme	Non Informative Cluster, Non Informative Elements		✓		✓
	Informative Cluster, Non Informative Elements		✓	✓	✓
	Non Informative Cluster, Informative Elements		✓	✓	
	Informative Cluster, Informative Elements	✓			✓
	Weights and Scalings [a]	U, WU, WS1, WS2 WS1IS[b], WS2IS[b], Method C[b]	U, WU, WS1, WS2	U, WU, WS1, WS2	U, WU, WS1, WS2
Population and Sample Sizes	Cluster Population Size (K)	500	Unknown	1500	300
	Cluster Sample Size (k)	about 300	100	33, 99	35, 75
	# Population Elements per Cluster (N_k)	5, 10, 20, 50, 100	Unknown	100, 5	Random: 38 to 147
	# Sampled Elements per Cluster (n_k)	0.5 N_k	5, 20, 100	75, 5, 4	9, 38, $0.4N_k$, $0.1N_k$

Table 1: Summary of Previous Simulation Study Designs

[a] U = Unweighted, WU = Weighted Unscaled, WS1 = Weighted Scaled 1, WS2 = Weighted Scaled 2

[b] WS1IS = Weighted Scaled 1 Invariant Selection, WS2IS = Weighted Scaled 2 Invariant Selection. See Section 3.3 for more details

value.

When analyzing the effect of the weights on informative sampling, note that the undersampling of large random intercepts (i.e. undersample $|U_{0k}| \geq 1$) should cause the unweighted estimates of σ^2_{0k} to be too small and the undersampling of error terms greater than zero (i.e. $\epsilon_{ik} \geq 0$) should cause the unweighted estimate of β_0 to have negative bias.

As can be seen from Table 2, the unweighted estimate of β_0 is biased under all sample sizes. The weights reduce this bias, however it is not until the cluster population sizes are $N_k = 50$ that the bias becomes negligible (recall from Table 1 that the sample size is roughly half of the population size). The unweighted estimates of σ^2_{0k} are also biased. The effect of adding the weights is mixed for σ^2_{0k}. For the N_k=5, the bias is reduced by all the weights. For the other values of N_k, there is at least one weighting scheme that produces the same (or larger) bias than the unweighted estimate and there are some weighting schemes that appear to do well, however none of the weighting schemes eliminate the bias.

When analyzing the the differences in the scaling of the weights, recall that the scaling is to help correct the bias in the weighted unscaled estimates of the random effect variances. The weighted unscaled estimates of σ^2_{0k} have a positive bias. Both the scaled 1 and scaled 2 estimates appear to overcorrect this bias, resulting in negative bias for the corresponding weighted estimates of σ^2_{0k}, however the bias of the weighted scaled 2 estimates appear to be smaller than the bias in the weighted scaled 1 estimates. For the larger population sizes, $N_k = 20$ or 50, the weighted unscaled estimates do as well or better than the weighted scaled 2 estimates.

RHS use the sandwich estimator to compute the standard errors of the point estimates. To evaluate the variances, RHS simulate the model 1000 times when the cluster size was $N_k = 50$ (while sampling 1/2 of the elements per cluster) and computed confidence intervals. The coverage of the RHS 95% confidence intervals created with the sandwich estimate variances range from 94.1% to 94.7% for the fixed effects, and is 92.4% for σ^2_{0k}.

	Design	Weighting Scheme	β_0	β_1	β_2	σ^2_{0k}
Simulation 1	Clusters:	unweighted	bias (0.60)	bias (0.08)	bias (0.06)	bias (0.61)
N_k=5	Undersample $\|U_k\| > 1$	weighted unscaled	unbiased	bias (0.19)	bias (0.22)	bias (0.47)
	Elements:	weighted scaled 1	bias (0.32)	unbiased	bias (0.06)	bias (0.42)
	Undersample $\epsilon_{ik} > 0$	weighted scaled 2	bias (0.25)	unbiased	unbiased	bias (0.30)
Simulation 2	Clusters:	unweighted	bias (0.63)	bias (0.13)	bias (0.14)	bias (0.23)
N_k=10	Undersample $\|U_k\| > 1$	weighted unscaled	bias (0.04)	bias (0.06)	bias (0.11)	bias (0.19)
	Elements:	weighted scaled 1	bias (0.17)	bias (0.09)	bias (0.09)	bias (0.60)
	Undersample $\epsilon_{ik} > 0$	weighted scaled 2	bias (0.12)	bias (0.06)	unbiased	bias (0.26)
Simulation 3	Clusters:	unweighted	bias (0.64)	bias (0.16)	bias (0.16)	bias (0.18)
N_k=20	Undersample $\|U_k\| > 1$	weighted unscaled	unbiased	bias (0.05)	bias (0.05)	bias (0.09)
	Elements:	weighted scaled 1	bias (0.09)	bias (0.06)	bias (0.05)	bias (0.30)
	Undersample $\epsilon_{ik} > 0$	weighted scaled 2	bias (0.06)	unbiased	unbiased	bias (0.17)
Simulation 4	Clusters:	unweighted	bias (0.65)	bias (0.18)	bias (0.18)	bias (0.13)
N_k=50	Undersample $\|U_k\| > 1$	weighted unscaled	unbiased	unbiased	bias (0.02)	bias (0.05)
	Elements:	weighted scaled 1	bias (0.04)	unbiased	bias (0.02)	bias (0.13)
	Undersample $\epsilon_{ik} > 0$	weighted scaled 2	unbiased	unbiased	unbiased	bias (0.06)

Table 2: RHS Simulation Design and Results

3.3 ASP Simulation Summary

Asparouhov (2006), denoted ASP, performed quite extensive simulations in his paper. These simulations vary the type of informative sampling, the intraclass correlation and the model being simulated. He uses many scalings for the weights, including unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2 estimation methods. ASP ran one simulation comparing the unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2 weights. He investigated the effect of the intra-class correlation on the weighted scaled 2 estimates and looked at a multilevel logistic regression with weighted scaled 2 estimates. The results of his simulations are summarized in Table 3.

For the informative sampling and intra-class correlation simulations, ASP uses the random intercept model,

$$y_{ik} = 0.5 + U_{0k} + \epsilon_{ik}, \quad U_{0k} \sim N(0, 0.5), \quad \epsilon_{ik} \sim N(0, 2) \tag{1}$$

where the population cluster size is 100, and the number of sampled individuals per cluster

Design	Recommended Weighting Scheme	Notes
Informative Sampling	Alternate method where all weights are scaled by the estimated population size divided by the sample size	Weighted Scaled 1 and Weighted Scaled 2 both also did well. All methods do best when cluster size is large or informativeness is weak.
Intra-Class Correlation	Weighted Scaled 2	Only Weighted Scaled 2 was analyzed. It was confirmed that when the ICC is small all parameters exhibit more bias.
Multi-Level Logistic	Weighted Scaled 2	Only Weighted Scaled 2 method was analyzed. Bias increases as sample size decrease and informativeness increases.

Table 3: ASP Simulation Design and Results

is 5, 20 and 100. The population sizes are unknown. The informative sampling simulation samples individuals proportional to $(1+\exp\{-\frac{y_{ik}}{\alpha}\})^{-1}$, where the level of informativeness is determined by the constant α. With this sampling, larger values of y_{ik} are oversampled, which means that elements with larger random intercepts, U_{0k} and/or larger random errors, ϵ_{ik} are oversampled. I would expect to see that the variances of U_{0k} and ϵ_{ik} to be underestimated, with the variance of U_{0k} to be affected more by the informative sampling.

ASP's results are as expected. None of the weighting methods performed well on all three parameters (the intercept and the variances of U_{0k} and ϵ_{ik}) unless the level of informativeness was small, or the sample size was large (over 100). The weighting methods generally correct for the informative sampling in the fixed effects, however for the random effects it takes sample sizes of 100 to see corrections.

When analyzing the differences in the scalings of the weights, the best weighting to use is not clear. For the informative sampling simulation, weighted scaled 1 , weighted scaled 2 and ASP's method C (where the scaling for the weights is the estimated population size divided by the sample size, $\sum_{ik} w_{ik}/\sum_k n_k$) all perform equivalently.

ASP uses the sandwich estimator to compute the standard errors of the point estimates. He reported the coverage of the corresponding 95% confidence intervals for all estimates.

Finally, ASP also performs simulations that verify that the bias of the variance components increase as the ICC increases. ASP also estimates a multi-level logistic regression model with a random effect and concludes that the bias increases as the sample size decreases and informativeness increases.

3.4 KG Simulation Summary

KG are primarily concerned with method of moment estimators, however for the random intercept model with no covariates, the method of moment estimators match the weighted MLE estimates. They ran simulations using a random intercept model,

$$y_{ik} = 1 + U_{0k} + \epsilon_{ik}, \quad U_{0k} \sim N(0,1), \quad \epsilon_{ik} \sim N(0,1).$$

The simulations contain 1500 population clusters (K), of which 33 or 99 are sampled (k). The population cluster sizes (N_k) are 100 and 5, and sample cluster sizes (n_k) are 75 and 5 ($N_k = 100$), and 4 ($N_k = 5$). The goal of KG's method is to improve the small sample properties of the weighted estimators. The bias from the KG simulations are summarized in Table 4. They did not report the estimates of β_0. It is unknown how many simulations are averaged for these means, and the variances of these means were not reported for these simulations.

When analyzing the effect of the weights on informative sampling, their simulations show that the bias is effectively removed with their weighted estimates, even with small sample sizes ($K = 1500, k = 33, N_k = 100, n_k = 5$). This is impressive; however the KG method uses additional information that the other methods do not use (the bivariate, trivariate and quadvariate inclusion weights, $w_{ij|k}, w_{ijl|k}, w_{ijlm|k}$).

KG did not compare their weights in this simulation to unweighted, weighted unscaled, weighted scaled 1 or weighted scaled 2 estimates.

KG use the jackknife estimator for design based survey sampling to estimate the vari-

Design	Sampling	σ^2_{0k}	σ^2_ϵ
Clusters:	$k = 33$ or $99, K = 1500$	0.03	unbiased
Undersample $\lvert U_k \rvert > 0.6745$	$n_k = 75, N_k = 100$	0.01	unbiased
Elements:SRS	$k = 33$ or $99, K = 1500$	0.01	0.01
	$n_k = 5, N_k = 100$	unbiased	unbiased
	$k = 33$ or $99, K = 1500$	0.01	0.01
	$n_k = 4, N_k = 5$	0.01	unbiased
Clusters:	$k = 33$ or $99, K = 1500$	0.01	unbiased
Census	$n_k = 75, N_k = 100$	unbiased	unbiased
Elements: SRS	$k = 33$ or $99, K = 1500$	unbiased	0.01
Undersample $\lvert \epsilon_{ik} \rvert > 0.6745$	$n_k = 5, N_k = 100$	unbiased	unbiased
	$k = 33$ or $99, K = 1500$	unbiased	unbiased
	$n_k = 4, N_k = 5$	0.01	unbiased

Table 4: KG Simulation Design and Results

ances of their point estimates. They did not compute the variances for the simulation summarized in Table 4.

3.5 PSHGR Simulation Summary

Stapleton (2002) and Huang and Hidiroglou (2003) conducted simulation studies using the PSHGR method. Their results are not described here as they support the results from the PSHGR simulation study, which is described next. PSHGR ran three simulation studies varying the level of informative sampling in a random intercept model with no covariates,

$$y_{ik} = 1 + U_{0k} + \epsilon_{ik}, \quad U_{0k} \sim N(0, 0.2), \quad \epsilon_{ik} \sim N(0, 0.5).$$

For each simulation there were 300 population clusters and 35 were sampled. They also ran simulations where 75 clusters were sampled, though they did not show those results and indicated that the results were similar. The number of population elements per cluster, N_k, was random and bounded between 38 and 147 with a mean of 80. They varied the number of sampled elements, n_k, between 38, $0.4 \times N_k$, 9 and $0.1 \times N_k$. The simulations contained different combinations of informative cluster sampling, non-informative cluster

	Design	Weighting Scheme	β_0	σ^2_{0k}	σ^2_ϵ
Simulation 1	Clusters: PPS	Unweighted	biased	varied*	biased
	where Size = U_k	Weighted Unscaled	unbiased	varied*	unbiased
	Elements	Weighted Scaled 1	unbiased	varied*	varied*
	Undersample $\epsilon_{ik} > 0$	Weighted Scaled 2	unbiased	varied*	unbiased
Simulation 2	Clusters: PPS	Unweighted	biased	varied*	unbiased
	where Size = U_k	Weighted Unscaled	unbiased	varied*	varied*
	Elements: SRS	Weighted Scaled 1	unbiased	unbiased	unbiased
		Weighted Scaled 2	unbiased	unbiased	unbiased
Simulation 3	Clusters: PPS	Unweighted	unbiased	unbiased	unbiased
	where Size = N_k	Weighted Unscaled	unbiased	varied*	varied*
	Elements: SRS	Weighted Scaled 1	unbiased	unbiased	unbiased
		Weighted Scaled 2	unbiased	unbiased	unbiased

* bias varied according to sample size

Table 5: PSHGR Simulation Design and Results

sampling, informative individual sampling and non-informative individual sampling. The simulations and results are in Table 5. PSHGR did not provide estimates of variances for all the simulated scenarios. As a rule of thumb, in Table 5, I marked an estimate as biased if the average over the iterations deviates more than 10% from the true value.

When analyzing the effect of weights on informative sampling, note that sampling clusters proportional to U_{0k} should introduce bias in the unweighted estimates of β_0 and σ^2_{0k}. Sampling of individuals proportional ϵ_{ik} should introduce bias in the estimate of β_0 and σ^2_ϵ. PSHGR found that when there is informative sampling of clusters, the expected biases appear. The use of the weights compensates for the bias in the estimate of β_0, however the effect of the weights on the estimates of the variance components varies according to the sample size.

When analyzing the differences in the scaling of the weights, PSHGR tentatively recommended weighted scaled 2 estimates. The bias of the weighted unscaled estimates varied according to the sampling size for all of the sampling scenarios. The weighted scaled 1 and weighted scaled 2 estimates performed better when there was less informative sampling.

PSHGR estimate the variances of the point estimates with design-based methods. They did not estimate the variances in their simulation study.

3.6 Summary

From the initial simulations, KG appears to have the lowest bias. The methods need to be compared using the same simulated conditions to get an accurate comparison. Because KG requires the higher order (i.e. bivariate, trivariate and quadvariate) conditional weights, they use more information than the other methods, which may result in better estimates. In reviewing the RHS, ASP and PSHGR simulations, it is not clear which method provides better results.

RHS and ASP found that the coverage levels of the confidence intervals based on the sandwich estimator were very close to the intended coverage. PSHGR provided simulation estimates of variances based on the design based variance estimator but did not evaluate their performance.

None of the four papers investigating the weights contained simulations with model misspecification.

All of the simulations showed that adding weights to the analysis helped compensate for the bias due to informative sampling. The informative sampling in all of the simulations was directly based on either the value of the random effect, U_{0k}, the random error, ϵ_{ik} or the value of the outcome variable, y_{ik}.

RHS and PSHGR both tentatively recommend the weighted scaled two estimates when there is informative sampling. ASP appears to favor weighted scaled 2 weights, as those are the weights used to evaluate the intra-class correlation and the multi-level logistic regression. RHS, ASP and PSHGR found the bias decreases as the sample size increases for all weighting schemes.

4 Format of New Simulation Results

Figure 1 contains a sample of the format of the new simulation results presented in this dissertation; it is the first row of Figure 4, which appears later in Section 5 (as do most of the other tables and equations referred to here). The caption on the figure specifies the name and simulation number which correspond to the columns in the summaries in Tables 6 and 7. Also included in the caption is the equation number of the generating equation. For Figure 1, a summary of the simulation is in the "Mis Ran 5" column of Table 6. The generating model for Figure 1 is in Equation 10. To the left of the plots is the estimated model for the variables in that row. In Figure 1, the estimated model is in Equation 11.

Each of the panels in Figure 1 represents a possible parameter in the estimated model. The parameter name is in bold at the top of the plot. Next to the parameter name (in parenthesis) is the variable associated with that parameter, if applicable. Below the parameter name is the range for the parameter. If there is no range (and no plot) printed, then that parameter was not estimated in this model, such as the σ^2_{0k} parameter in Figure 1. The solid vertical line indicates the true value of the parameter as it is in the generating model. The horizontal lines represent the 0.025 to 0.975 empirical quantiles over the simulation replicates for that set of estimates. The circle in the horizontal line represents the average of the estimates. Each plot contains eight horizontal line plots.

Figure 1: Results for Misspecification of Random Variables, Simulation Set 5
Generated Model - Equation 10

The red horizontal line plots represent the PSHGR simulations, and the black horizontal line plots represent the RHS simulations. Each of the horizontal line plots has a caption, such as "R - S2 (70/86/100)". The first term in the caption is either an R (for RHS) or a P (for PSHGR) representing the estimation method used. The second term represents the type of weighting estimation used, either S2 for weighted scaled 2 estimates, S1 for weighted scaled 1 estimates, WU for weighted unscaled estimates or UN for unweighted estimates. Finally, there are three numbers listed. The first number is the number of estimated confidence intervals that contained the true parameter value. These confidence intervals are computed as the point estimate for the given simulation plus or minus 2 times the standard error. The standard error is computed using a sandwich estimator for RHS and a design based estimator for PSHGR, as they did in their papers. The second number represents the number of the iterations where the variance was able to be computed. For RHS, the code to run the simulations is not always able to estimate variances for the estimated point estimates (the estimate of the Hessian is sometimes numerically unstable). Finally, the third number in the caption is the number of iterations where the point estimates are able to be estimated. If the number of iterations is less than 100 for RHS, then it means that some iterations did not converge for any of number of quadrature points between 15 and 35. If the number of iterations is less than 100 for PSHGR, then it means that the iterative generalized least squares algorithm did not converge within 500 iterations.

For example, the RHS weighted unscaled estimates of σ_ϵ^2 are in the fifth plot from the left. The caption on the horizontal line plot is "R - WU (48/95/100)". This means that of the 100 iterations, all 100 of them are able to produce weighted unscaled estimates of the σ_ϵ^2 parameter. Of the 100 iterations that are able to produce point estimates, 95 of them are able to produce estimates of the variances. Of the 95 iterations able to produce estimates of the variance, 48 of the estimated confidence intervals contained the true parameter value of 0.5. Thus, the estimated coverage of the 95% RHS confidence intervals (as computed with the sandwich estimator variance) is 48/95=50.5%. The horizontal line represent the 0.025 and 0.975 quantiles of the 100 point estimates generated. The average of the 100 estimates is about 0.4, representing a bias of approximately 0.1. When comparing this to the RHS unweighted estimate of σ_ϵ^2, it is clear that there is a smaller spread for the unweighted variances than the weighted variances. In addition, the unweighted estimates are approximately unbiased, and the 95% confidence interval covers the true parameter values 90/100=90% of the time.

5 New Simulation Results

The new simulations presented below confirm and expand upon the previously published results. The new simulations that are performed refer to the RHS method, published concurrently with ASP, as the RHS method because the software used to run the simulations was written by Rabe-Hesketh and Skrondal (see `www.gllamm.org`). In addition, the simulations by KG are summarized in this chapter, however I did not perform further simulations of their method as most analysts will not have the joint and quadruple conditional weights ($w_{ij|k}$ and $w_{ijlm|k}$) weights needed to implement their method.

There are a total of 12 simulation sets, broken into 4 categories: 1) Misspecification of the Fixed Effects, 2) Misspecification of the Random Effects, 3) Misspecification of Stratification Layers and 4) Misspecification of Clustering Layers. The simulation sets are summarized in Tables 6 and 7. Each simulation category contains model misspecification and/or informative sampling. The definitions of sampling completely at random, sampling at random, sampling not at random (or informative sampling) are analogous to the similar missing data terminology from Little and Rubin (2002) and are used to describe the extent of informative sampling in the simulations.

The summary of each simulation reflects on the conclusions from this chapter.

1. The PSHGR and RHS point estimates are very similar. The differences in the point

		Mis Fix[a] 1	Mis Fix[a] 2	Mis Fix[a] 3	Mis Fix[a] 4	Mis Ran[b] 5	Mis Ran[b] 6	Mis Ran[b] 7	Mis Ran[b] 8
Generating Model	Random Intercept Model: $Y_{ik} = 1 + U_{0k} - 2x_{1k} + 2x_{2ik} + \epsilon_{ik}$, $U_{0k} \sim N(0, 0.2)$, $\epsilon_{ik} \sim N(0, 0.5)$	✓	✓	✓	✓				
	Random Slope Model: $Y_{ik} = 1 + (-2 + U_{1k})x_{1k} + 2x_{2ik} + \epsilon_{ik}$, $U_{0k} \sim N(0, 0.2)$, $\epsilon_{ik} \sim N(0, 0.5)$					✓	✓		
	Random Slope Model: $Y_{ik} = 1 + -2x_{1k} + (2 + U_{2k})x_{2ik} + \epsilon_{ik}$, $U_{0k} \sim N(0, 0.2)$, $\epsilon_{ik} \sim N(0, 0.5)$							✓	✓
Estimated Model	Same as generated	✓	✓	✓	✓	✓	✓	✓	✓
	Missing x_{1k}	✓	✓	✓	✓				
	Missing x_{2ik}	✓	✓	✓	✓				
	Missing U_{1k} added random intercept U_{0k}					✓	✓		
	Missing U_{2k} added random intercept U_{0k}							✓	✓
Sampling Scheme	Cluster Sample PPS N_k, element sampling PPS independent variable	✓				✓		✓	
	Cluster Sample PPS N_k, element sampling PPS x_{2ik}		✓						
	Cluster Sample PPS x_{1k}, element sampling PPS independent variable			✓					
	Cluster Sample PPS x_{1k}, element sampling PPS x_{2ik}				✓				
	Cluster Sample PPS U_{1k}, element sampling PPS independent variable						✓		
	Cluster Sample PPS U_{2k}, element sampling PPS independent variable								✓

Table 6: Simulation Designs for the Misspecification of Fixed and Random Effects

[a]Mis Fix = Misspecification of Fixed Effects

[b]Mis Ran = Misspecification of Random Effects

		Mis Strat[c] 9	Mis Strat[c] 10	Mis Strat[c] 11	Mis Clust[d] 12
Generated Model	Generated Model: Random Intercept with necessary adjustments reflecting the sampling design	✓	✓	✓	✓
Estimated Model	Estimated Model: Random Intercept with necessary adjustments reflecting the sampling design	✓	✓	✓	✓
Generated Layers	Stratified / Clustered	✓			
	Clustered / Stratified		✓		
	Stratified / Clustered / Stratified			✓	
	Cluster 1/ Cluster 2				✓
Estimated Layers	Stratified / Clustered	✓		✓	
	Clustered / Stratified		✓	✓	
	Clustered	✓	✓	✓	
	Clustered 1				✓
	Clustered 2				✓
Sampling Scheme	Clusters Sampling PPS Size, Element Sampling PPS independent variable	✓	✓	✓	✓
	Clusters Sampling PPS U, Element Sampling PPS independent variable	✓	✓		

Table 7: Simulation Designs for the Misspecification of Stratification and Clustering Layers

[c]Mis Strat=Misspecification of Stratification Layers
[d]Mis Clust= Misspecification of Clustering Layers

estimates are due to numerical instabilities in the estimation procedures.

2. The sandwich estimator, used by RHS , is a better estimator of the variance of the point estimates than the design-based variance estimator used by PSHGR. However, the sandwich estimator is not as numerically stable since computation of the Hessian is not always possible. The PSHGR design-based variance estimator appears reasonable when the model is correctly specified, however the estimates are sometimes too large when the model is misspecified, especially for the variance components.

3. When there is model misspecification that does not induce informative sampling, weighted estimates do not reduce bias of the estimators.

4. When there is informative sampling, the weighted estimators do reduce the bias of the point estimates, though they do not eliminate it.

5. The unweighted estimate has the smallest variance. When there is informative sampling, the unweighted estimates are biased. The weighted unscaled estimate corrects the bias in the fixed effects, but produces bias in the random effects. The scaled 1 weightings remove the bias in the fixed effects, and usually reduces (or overcorrects) for the weighted unscaled bias in the random effects. The scaled 2 weightings remove the bias in the fixed effects and are in between the weighted unscaled and weighted scaled 1 bias in the random effects. There are some cases where the scaled 1 estimates are more biased in the same direction as the weighted unscaled estimates. In these cases, the weighted scaled 2 estimates are still between the weighted unscaled and weighted scaled 1 weights. The variation of the estimates across the 100 iterations are somtimes similar for all estimates (weighted or unweighted). When the variation across the 100 iterations varies by the weighting, then the smallest variation is in the unweighted estimates, followed by the weighted scaled 1, weighted scaled 2 and unweighted estimates.

5.1 Misspecification of Fixed Effects - Non-Informative Sampling - Simulation Set 1

A summary of this simulation set is in the "Mis Fix 1" column of Table 6. The generating model is a random intercept model,

$$y_{ik} = 1 + U_{0k} - 2x_{1k} + 2x_{2ik} + \epsilon_{ik}, \quad U_{0k} \sim N(0, 0.2), \quad \epsilon_{ik} \sim N(0, 0.5), \tag{2}$$

where $x_{1k} \sim N(3, 9)$ and $x_{2ik} \sim N(1, 25)$. There are 300 population clusters, with a random uniform number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The sampling of clusters is proportional to the magnitude of the population cluster size, N_k. Sampling of individuals within clusters is proportional to an independently generated random variable assigned to each element[1]. There are three estimated models in this simulation set. One matches the generated model, one removes the fixed effect for x_{1k} , and one removes the fixed effect for x_{2ik},

$$y_{ik} = \beta_0 + U_{0k} + \beta_1 x_{1k} + \beta_2 x_{2ik} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{3}$$

$$y_{ik} = \beta_0 + U_{0k} + \beta_2 x_{2ik} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{4}$$

$$y_{ik} = \beta_0 + U_{0k} + \beta_1 x_{1k} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon). \tag{5}$$

The sampling scheme is sampling completely at random for all three estimated models.

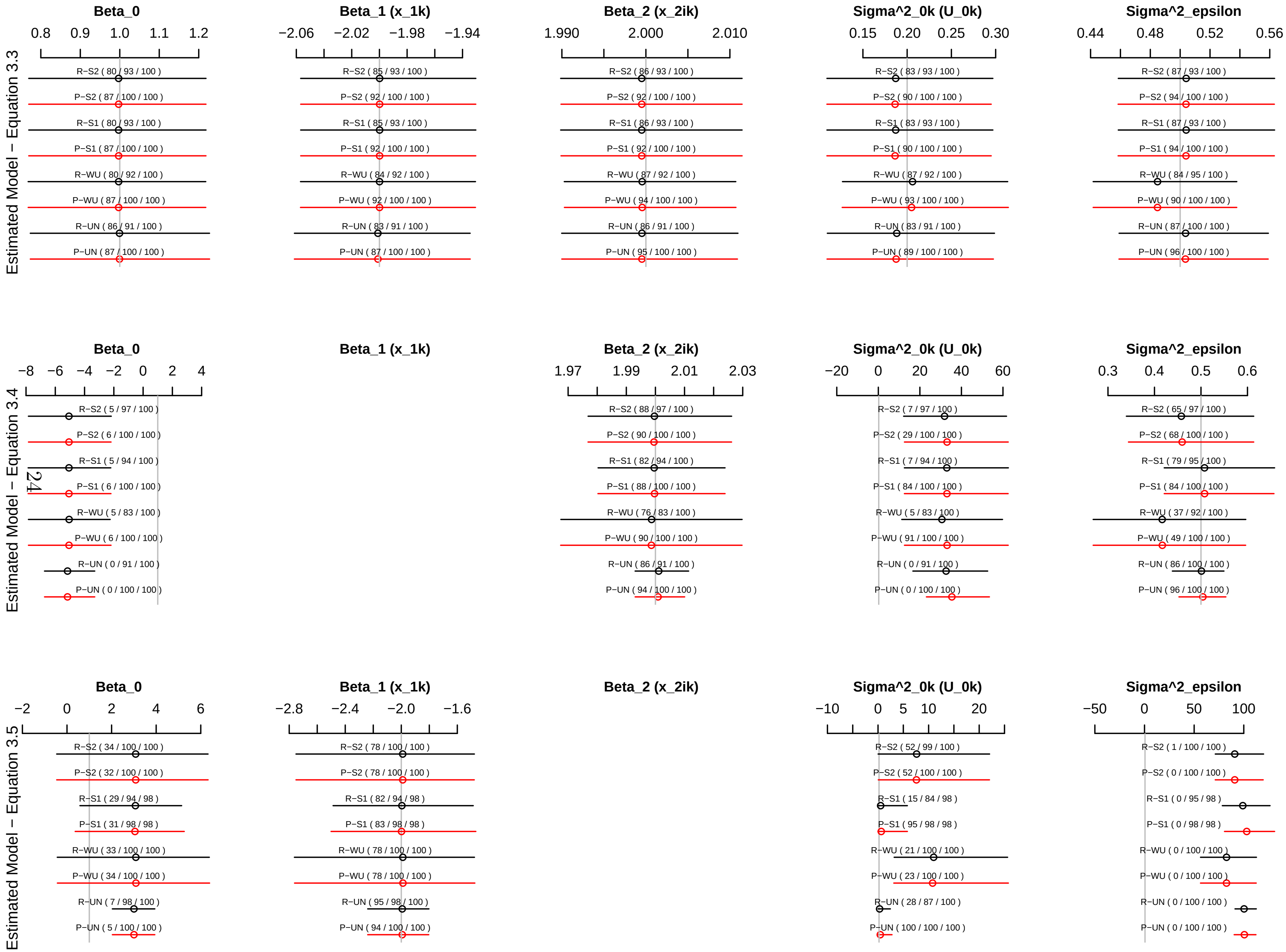
5.1.1 Summary

The results from this simulation set are in Figure 2. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method generally matched the

[1]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5, 5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

estimation using the RHS method. Some differences between PSHGR and RHS appear in Figure 2. The PSHGR unweighted estimates of σ^2_{0k} from Equation 4 have a larger mean and a larger 0.025 empirical quantile than RHS.



24

Figure 2: Results for Misspecification of Fixed Effects, Simulation Set 1Generated Model - Equation 2

The PSHGR weighted unscaled estimate of σ^2_{0k} from Equation 4 has a larger mean and larger 0.025 and 0.975 empirical quantiles than RHS. Finally, the PSHGR weighted scaled 1 estimate of σ^2_ϵ has a larger mean and larger 0.025 and 0.975 quantiles than RHS. These differences (and smaller differences not visible in Figure 2) are due to numerical instabilities in the RHS and PSHGR estimations, and are described in detail in Section 8.1.

When analyzing the coverage of the confidence intervals, look at the simulation where the estimating model is from Equation 3, which matches the generating model and has the least bias. The coverage from the RHS 95% confidence interval coverage varies between 85% and 95% in the fixed effects and between 83% and 87% in the variance components. For PSHGR, the 95% confidence interval coverage varies between 87% and 95% for the fixed effects and between 89% to 96% in the variance components. RHS produced sandwich estimates for the variance for between 83 and 100 of the 100 iterations for the estimates in Figre 2. The PSHGR estimates of the variance of σ^2_{0k} were quite large in estimated models from Equations 4 and 5, causing the confidence interval coverage to be much larger than the coverage from RHS. This may indicate a problem with the variance estimator for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained.

The second and third estimated models from Equations 4 and 5 contained model misspecification. When a covariate was included in the generating model but not the estimating model, a model misspecification bias was found in all weighting methods. The removal of a fixed covariate caused the intercept to change by the mean of the missing covariate times its associated parameter. The variance of the missing covariate moved into the intercept variance (if it was a cluster covariate) or the random error variance (if it was in individual covariate). It is possible that the missing covariate could affect both variance estimates if the covariate was an individual covariate whose mean varied across clusters. See Section 8.1 for more details for this simulation. The various weighting methods did not help against model misspecification bias.

These simulations did not contain any informative sampling, so there was no informative sampling bias.

All weighting methods provided similar mean estimates of the β coefficients. The 0.024 and 0.975 quantiles over the simulation runs sometimes vary according to weighting scheme. When the model is correctly specified, all estimates (weighted and unweighted) have similar spread across the simulations. When the model is misspecified, the spreads sometimes differ. When they do, the unweighted has the smallest spread, followed by the weighted scaled 1, weighted scaled 2 and weighted unscaled estimates. There is a difference in the weighting schemes with the estimation of the variance components. The weighted unscaled estimates have a bias, the weighted scaled 1 estimate compensates (or overcompensates) for the weighted unscaled bias and the weighted scaled 2 bias is between the weighted scaled 1 and the weighted unscaled bias. How close the weighted scaled 2 bias is to the weighted scaled 1 bias appears to vary. When the model is correctly specified, the weighted scaled 1 and weighted scaled 2 estimates of the variance components (both σ_{ϵ}^2 and σ_{0k}^2) are close. When there is model misspecification, the weighted scaled 2 estimates appear to be balanced in between the weighted scaled 1 and the weighted unscaled estimates.

5.2 Misspecification of Fixed Effects - Partially Informative Sampling - Simulation Sets 2 and 3

A summary of these simulations sets are in the "Mis Fix 2" and "Mis Fix 3" columns of Table 6. The generating model for both simulation sets is a random intercept model,

$$y_{ik} = 1 + U_{0k} - 2x_{1k} + 2x_{2ik} + \epsilon_{ik} \quad U_{0k} \sim N(0, 0.2), \quad \epsilon_{ik} \sim N(0, 0.5).$$

where $x_{1k} \sim N(3, 9)$ and $x_{2ik} \sim N(1, 25)$. The population has 300 clusters, each with a random number of units per cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The three estimated models are

$$\begin{aligned}
y_{ik} &= \beta_0 + U_{0k} + \beta_1 x_{1k} + \beta_2 x_{2ik} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_{\epsilon}) \\
y_{ik} &= \beta_0 + U_{0k} + \beta_2 x_{2ik} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_{\epsilon}) \\
y_{ik} &= \beta_0 + U_{0k} + \beta_1 x_{1k} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_{\epsilon}).
\end{aligned}$$

5.2.1 Result Description for Misspecification of Fixed Effects – Simulation Set 2

For Misspecification of Fixed Effects - Simulation Set 2, the sampling of clusters is proportional to the magnitude of the population cluster size, N_k. The sampling of individuals in a cluster is proportional to the magnitude of the individual level covariate x_{2ik}. The simulation is sampling at random when the covariate x_{2ik} is included in the estimating model, and informative sampling when the estimating model did not contain the covariate. When the model did contain the covariate x_{2ik}, then the estimation behaved exactly as in Misspecification of Fixed Effects - Simulation Set 1, where there is no informative sampling. When the model did not contain the x_{2ik} covariate, the estimation behaved exactly as in Misspecification of Fixed Effects - Simulation Set 4, where there is informative sampling. For space considerations, the results for Misspecification of Fixed Effects - Simulation Set

2 were not presented here.

5.2.2 Result Description for Misspecification of Fixed Effects - Simulation Set 3

For Misspecification of Fixed Effects - Simulation Set 3, the sampling of clusters is proportional to the magnitude of the cluster level covariate x_{1k}. The sampling of individuals is proportional to an independently generated random variable assigned to each element[2]. The simulation was sampling at random when the variable x_{1k} was included in the estimating model, and informative sampling when the estimating model did not contain the covariate x_{1k}. When the model did contain the covariate x_{1k}, then the estimation behaved exactly as in Misspecification of Fixed Effects - Simulation Set 1, where there is no informative sampling. When the model did not contain the x_{1k} covariate, the estimation behaved exactly as in Misspecification of Fixed Effects - Simulation Set 4, where there is informative sampling. For space considerations, the results for Misspecification of Fixed Effects - Simulation Set 3 were not presented here.

[2]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5, 5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

5.3 Misspecification of Fixed Effects - Informative Sampling - Simulation Set 4

A summary of this simulation set is in the "Mis Fix 4" column of Table 6. The generating model is a random intercept model,

$$y_{ik} = 1 + U_{0k} - 2x_{1k} + 2x_{2ik} + \epsilon_{ik} \quad U_{0k} \sim N(0, 0.2), \quad \epsilon_{ik} \sim N(0, 0.5), \tag{6}$$

where $x_k \sim N(3, 9)$ and $x_{ik} \sim N(1, 25)$. There are 300 population clusters, with a random uniform number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The sampling of clusters is proportional to the magnitude of the cluster covariate, x_{1k}. Sampling of individuals is proportional to the magnitude of the individual covariate, x_{2ik}. The three estimated models are

$$y_{ik} = \beta_0 + U_{0k} + \beta_1 x_{1k} + \beta_2 x_{2ik} + \epsilon_{ik}, \quad U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{7}$$

$$y_{ik} = \beta_0 + U_{0k} + \beta_2 x_{2ik} + \epsilon_{ik}, \quad U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{8}$$

$$y_{ik} = \beta_0 + U_{0k} + \beta_1 x_{1k} + \epsilon_{ik}, \quad U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{9}$$

The simulation is sampling at random when both the x_{1k} and x_{2ik} covariates are included in the estimating model, and informative sampling when the estimating model does not contain either (or both) of the covariates.

5.3.1 Summary

The results from this simulation set are in Figure 3. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. There are some differences between the PSHGR and RHS estimates, but they are not large enough to be seen in Figure 3. See Section 8.1 for more details.

The coverage of the confidence intervals between RHS and PSHGR are similar for the estimated model in Equation 7. The RHS 95% confidence intervals for the β coefficients are between 73% and 97% and for the variance components they are between 58% to 87% . The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 7 are between 76% and 97% and for the variance components they are between 56% and 88%. The major difference between PSHGR and RHS is that PSHGR can compute the variances of the point estimates in all the simulation runs for all the parameters, whereas RHS computes the variances between 87% and 100% of the simulation runs. In addition, the PSHGR confidence intervals for σ_{0k}^2 in the estimated model from Equation 8 have larger coverage than expected, especially for the weighted unscaled estimate. This may indicate a problem with the variance computation for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained.

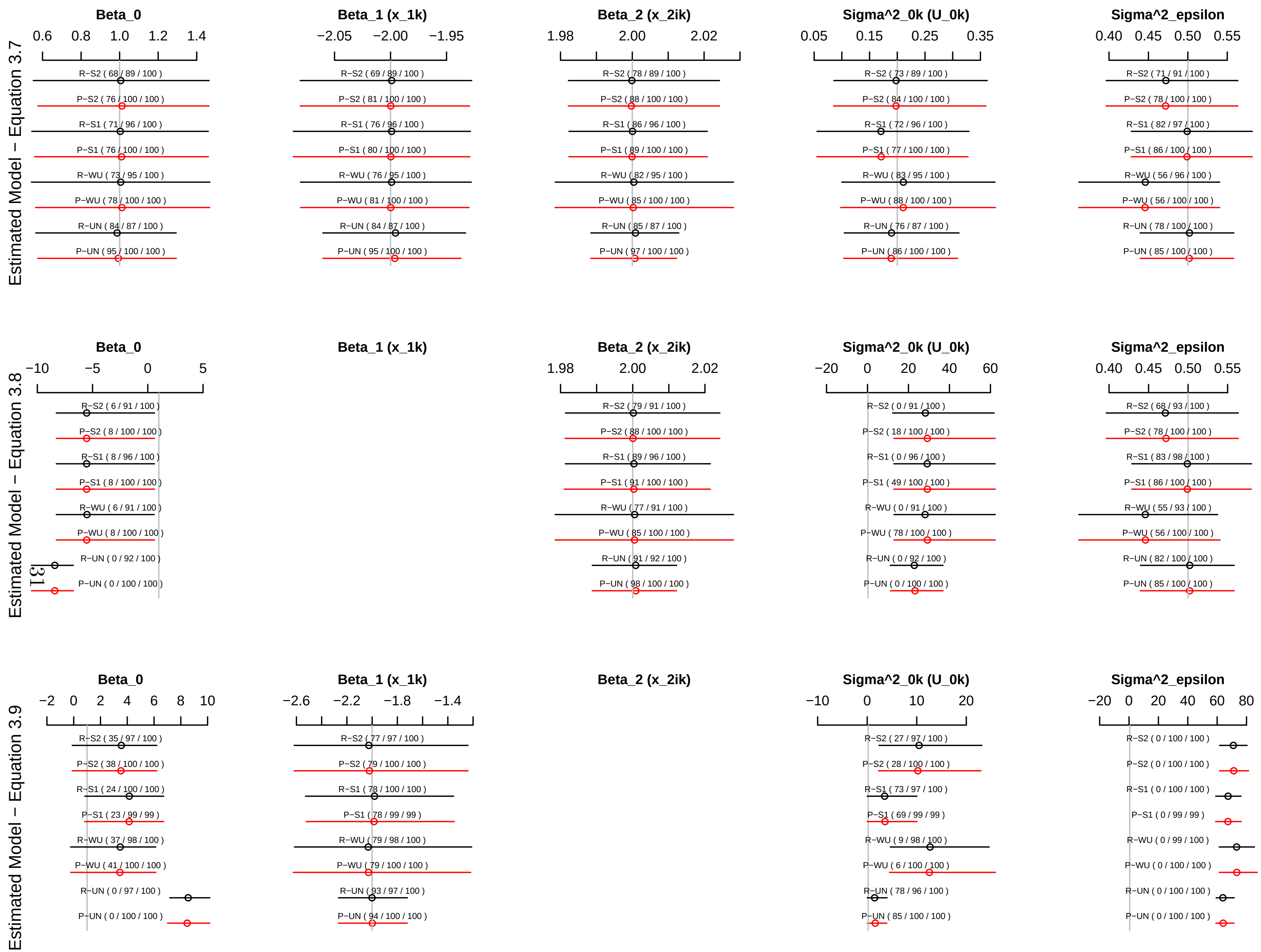


Figure 3: Results for Misspecification of Fixed Effects, Simulation Set 4Generated Model - Equation 6

The second and third estimated models contain model misspecification. The estimated models from Equations 8 and 9 contain model misspecification that induces informative sampling. For the estimated model defined in Equation 8, the weighted estimates of the intercept are near -5, as the are in Figure 2 under the estimated model in Equation 4 where there is no informative sampling. This is not near the true value of 1. This difference in the estimates is due to the model misspecification that is not related to informative sampling. A similar trend is seen in the estimate of the intercept from the estimated model in Equation 9. The weighted methods do not compensate for the model misspecification bias.

The second and third estimated models contain informative sampling. The informative sampling bias can be seen by comparing the unweighted estimates to the weighted estimates for β_0 from estimating models in Equations 8 and 9. It can also be seen in the estimates for σ^2_{0k}, but it is not so obvious. The unweighted estimate of σ^2_{0k} is the same size or larger than the weighted unscaled estimate of σ^2_{0k} in Figure 2 under the estimating model in Equation 4 where there is no informative sampling. However, the unweighted estimate of σ^2_{0k} is smaller than the weighted unscaled estimate of σ^2_{0k} in Figure 3 under the estimating model in Equation 8 where there is informative sampling. The same can be seen under estimated models in Equations 5 and 9, however it is not so clear since these estimates are against the constraint that $\sigma^2_{0k} \geq 0$. See section 8.1 for more details. Note that the weights do not fully compensate for the informative sampling bias, as can be seen by comparing the estimates of σ^2_ϵ from the estimated model in Equation 9 to the estimates of σ^2_ϵ in Figure 2 under the estimated model in Equation 5. The addition of the weights helped to compensate for the informative sampling.

All weighting methods generally provide similar point estimates and ranges for the β coefficients. The exception is that the spread of the weighted unscaled estimates of β_2 appear to be larger. The estimates for β_0 from the estimated model in Equation 9 vary more then the other β coefficients. For both σ^2_ϵ and σ^2_{0k}, there is some bias in the unweighted estimates. The weighted unscaled estimates have a larger bias, the weighted scaled 1 estimate compensates (or overcompensates) for the weighted unscaled bias and the weighted scaled 2 bias is in between the weighted scaled 1 and the weighted unscaled bias. Note that the weighted scaled 2 estimates of σ_ϵ^2 and σ_{0k}^2 when the model is correctly specified are further from the weighted scaled 1 estimates than in Figure 2. This indicates that the scaled 2 weights may help with estimation of the variance components under sampling at random. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.4 Misspecification of Random Variables - Non-Informative Sampling - Simulation Set 5

A summary of this simulation set is in the "Mis Ran 5" column of Table 6. The generating model is a random slope model with the random slope on a cluster level covariate,

$$y_{ik} = 1 + (-2 + U_{1k})x_k + 2x_{ik} + \epsilon_{ik} \quad U_{1k} \sim N(0,1), \quad \epsilon_{ik} \sim N(0,0.5), \tag{10}$$

where $x_{1k} \sim N(3,9)$ and $x_{2ik} \sim N(1,25)$. There are 300 population clusters, with a random uniform number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The sampling of clusters is proportional to the magnitude of the population cluster size, N_k. Sampling of individuals within clusters is proportional to an independently generated random variable assigned to each element[3]. There are two estimated models in this simulation set. One matches the generated model, and one removes the random slope U_{1k} and adds a random intercept U_{0k},

$$y_{ik} = \beta_0 + (\beta_1 + U_{1k})x_k + \beta_2 x_{ik} + \epsilon_{ik}, \quad U_{1k} \sim N(0,\sigma^2_{1k}), \quad \epsilon_{ik} \sim N(0,\sigma^2_\epsilon) \tag{11}$$

$$y_{ik} = \beta_0 + \beta_1 x_k + \beta_2 x_{ik} + U_{0k} + \epsilon_{ik}, \quad U_{0k} \sim N(0,\sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0,\sigma^2_\epsilon) \tag{12}$$

The sampling scheme is sampling competely at random for both estimated models.

5.4.1 Summary

The results from this simulation set are in Figure 4. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. There are some differences between the PSHGR and RHS estimates, but they are not large enough to be seen in Figure 4. See Section 8.1 for more

[3]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5,5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

details.

The coverage of the confidence intervals of RHS and PSHGR are similar, with the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 11 are between 75% to 95%, and for the variance components they are between 50% and 90%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 11 are between 72% to 95%, and for the variance components they are between 49% and 96%. RHS was able to produce sandwich estimator variances for between 77% and 100% of the simulation runs, while PSHGR was able to produce design based estimator variances for 100% of the simulation runs. Again, the number of confidence intervals for PSHGR covering the true parameter appears larger than the RHS intervals, especially for the random effects and for the estimated model in Equation 12, where there is model misspecification. This may indicate a problem with the variance estimator for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained.

Figure 4: Results for Misspecification of Random Variables, Simulation Set 5
Generated Model - Equation 10

The second estimated model contains model misspecification. The random slope term is removed and a random intercept term is added. The random intercept variance contains the variance of the random slope term ($U_{1k}x_{1k}$), however there is some negative bias in the estimates. The expected variance of the random intercept is approximately 18, while the simulated means are between 13.5 and 16, see Section 8.1 for details. This is expected due to the low intra-class correlation, see Asparouhov (2006). None of the other estimates are affected by the model misspecification. Note that the weighted estimates do not appear to compensate for the model misspecification, though it is not entirely clear what compensating for model misspecification would mean in this example.

These simulations did not contain any informative sampling, so there was no informative sampling bias.

All weighting schemes provide similar point estimates and ranges for the β parameters. The exception is that the spread for the weighted unscaled estimate of β_2 is larger than the other weighted schemes. The variance of the unweighted estimates is smaller. The estimates of the random slope follow the trend that the weighted scaled 2 estimate is between the weighted unscaled and the weighted scaled 1. The bias doesn't quite follow the same pattern as the weighted scaled 1 estimates show more bias in the same direction as the weighted unscaled, as opposed to σ_ϵ^2 and σ_{0k}^2 where the weighted scaled 1 compensates for the bias in the weighted unscaled estimates. All the unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.5 Misspecification of Random Variables - Informative Sampling - Simulation Set 6

A summary of this simulation set is in the "Mis Ran 6" column of Table 6. The generating model is a random slope model, with the random slope on the cluster level covariate,

$$y_{ik} = 1 + (-2 + U_{1k})x_{1k} + 2x_{2ik} + \epsilon_{ik} \quad U_{1k} \sim N(0,1), \quad \epsilon_{ik} \sim N(0,0.5), \tag{13}$$

where $x_{1k} \sim N(3,9)$ and $x_{2ik} \sim N(1,25)$. There are 300 population clusters, with a random number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The sampling of clusters is proportional to the magnitude of the random effect, U_{1k}. Sampling of individuals within clusters is proportional to an independently generated random variable assigned to each element[4]. There are two estimated models in this simulation set. One matches the generated model, and one removes the random slope U_{1k} and adds a random intercept U_{0k},

$$y_{ik} = \beta_0 + (\beta_1 + U_{1k})x_k + \beta_2 x_{2ik} + \epsilon_{ik}, \quad U_{1k} \sim N(0,\sigma^2_{1k}), \quad \epsilon_{ik} \sim N(0,\sigma^2_\epsilon) \tag{14}$$

$$y_{ik} = \beta_0 + \beta_1 x_{1k} + \beta_2 x_{2ik} + U_{0k} + \epsilon_{ik}, \quad U_{0k} \sim N(0,\sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0,\sigma^2_\epsilon) \tag{15}$$

The sampling scheme is informative sampling at the cluster level.

5.5.1 Results Summary

The results from this simulation set are in Figure 5. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. The PSHGR estimate of β_0 under the estimated model in Equation 15 has a lower mean and a lower 0.025 quantile and a higher 0.975 quantile

[4] Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5,5)$. They were then sampled proportional to $(1+\exp(-a_{ik}))^{-1}$.

than the corresponding RHS estimate. This and other differences between PSHGR and RHS are described in more detail in Section 8.1. The coverage of the confidence intervals of RHS and PSHGR are similar, with the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 14 are between 10% to 96%, and for the variance components they are between 31% and 89%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 14 are between 11% to 95%, and for the variance components they are between 41% and 94%. RHS was able to produce sandwich estimator variances for between 80% and 100% of the simulation runs, while PSHGR was able to produce design based estimator variances for 100% of the simulation runs. In general, the number of PSHGR confidence intervals that cover the true parameter is larger than for RHS, especially when the model is misspecified as in the estimated model in Equation 15. This may indicate a problem with the variance computation for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained.

Figure 5: Results for Misspecification of Random Variables, Simulation Set 6
Generated Model - Equation 13

The second estimated model contained model misspecification. The random slope term was removed and a random intercept term was added. The random intercept variance contained the variance fo the random slope term ($U_{1k}x_{ik}$). None of the other estimates were affected by the model misspecification.

Both estimated models contain informative sampling. When the estimated and generated models match each other, the informative sampling causes the unweighted estimates of β_{1k} and σ^2_{1k} to be biased. All of the weighted estimates help to compensated for this informative sampling. When the random slope is removed from the model and a random intercept is added, the estimate of β_1 contained the same informative sampling bias in the unweighted estimate. The informative sampling bias of the σ^2_{1k} estimate is now reflected in the estimate of σ^2_{0k}. When comparing the unweighted estimate of σ^2_{0k} to the same estimate from the estimating model from Equation 12, it is clear that the unweighted estimate from the estimating model in Equation 15 is smaller. None of the other terms were affected.

All the weighted estimates performed similarly for the β coefficients. As in the previous simulations, for σ^2_ϵ and σ^2_{0k}, the weighted unscaled estimates are biased, the weighted scaled 1 estimates overcompensate for the bias, and the weighted scaled 2 estimates are in between. Note that unlike the previous simulation set that was non-informative, the pattern of the weights in the estimate of σ^2_{1k} follows the pattern of the other variance components. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.6 Misspecification of Random Variables - Non-Informative Sampling - Simulation Set 7

A sumary of this simulation set is in the "Mis Ran 7" column of Table 6. The generating model is a random slope model, where the random slope is on the individual level covariate,

$$y_{ik} = 1 - 2x_{1k} + (2 + U_{2k})x_{2ik} + \epsilon_{ik} \quad U_{2k} \sim N(0, 0.8), \quad \epsilon_{ik} \sim N(0, 0.5). \qquad (16)$$

where $x_{1k} \sim N(3, 9)$ and $x_{2ik} \sim N(1, 25)$. There are 300 population clusters, with a random uniform number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per cluster. The sampling of Clusters is proportional to the magnitude of the population cluster size, N_k. Sampling of individuals within clusters is proportional to an independently generated random variable assigned to each element[5]. There are two estimated equations in this simulation set. One matches the generated model, and one removes the random slope U_{2k} and adds a random intercept U_{0k},

$$y_{ik} = \beta_0 + U_{1k}x_{1k} + (\beta_2 + U_{2k})x_{2ik} + \epsilon_{ik}, \; U_{2k} \sim N(0, \sigma^2_{2k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \qquad (17)$$

$$y_{ik} = \beta_0 + \beta_1 x_{1k} + \beta_2 x_{2ik} + U_{0k} + \epsilon_{ik}, \; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \qquad (18)$$

This sampling scheme is sampling completely at random.

5.6.1 Summary

The results from this simulation set are in Figure 6. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. There are some differences between the PSHGR and RHS estimates, but they are not large enough to be seen in Figure 6. See Section 8.1 for more

[5] Each element was assigned a random variable $a_{ik} \sim$ Uniform$(-5, 5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

details. The coverage of the confidence intervals of RHS and PSHGR are mostly similar, with the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 20 are between 77% to 95%, and for the variance components they are between 49% and 94%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 17 are between 78% to 92%, and for the variance components they are between 59% and 98%. Note that the coverage of the σ^2_{2k} estimates for PSHGR (approximately 85/100) is much higher than the estimates of the coverage for RHS (approximately 45/95). The RHS coverages appear more accurate given the bias in the estimates. This may indicate a problem with the variance estimator for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained. RHS was able to produce sandwich estimator variances for between 92% and 100% of the simulation runs, while PSHGR was able to produce design based estimator variances for 100% of the simulation runs. In addition, for the estimated model in Equation 18, a simulation run did not converge for the RHS weighted scaled 2 estimates. The number of confidence intervals for PSHGR covering the true value fo σ^2_{2k} under the estimated model in Equation 17 are larger than the corresponding RHS intervals.

Figure 6: Results for Misspecification of Random Variables, Simulation Set 7
Generated Model - Equation 16

The second estimated model contains model misspecification. The variance from the dropped random slope is split between the estimated variance of the random intercept and the estimated variance of the random error, as expected from the description in Section 8.1. The estimates of β are not affected by the model misspecification. The addition of the weights does not help compensate for this model misspecification.

These simulations does not contain any informative sampling, so there is no informative sampling bias.

All the weighting schemes perform equivalently for the β estimates, except the weighted estimates of β_0 and β_1 with the unscaled weights have slightly larger variances. The weighting of the variance components follows the trend that the weighted unscaled estimates are biased, the weighted scaled 1 overcompensates for the bias, and the weighted scaled 2 estimates are between the weighted scaled 1 and the weighted unscaled estimates. An exception to this is the estimate of σ_ϵ^2 for the estimated model in Equation 18. Here we see that the weighted unscaled estimates are biased, and that the weighted scaled 1 estimates are more biased than the weighted scaled 1, with the weighted scaled two still between the weighted scaled 1 and the unweighted estimates. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread. The exception is in the estimated model in Equation 18 for the estimates of β_2 and σ_ϵ^2, where the scaled 1 estimates simulation spread is larger than the weighted scaled 2 spread.

5.7 Misspecification of Random Variables - Informative Sampling - Simulation Set 8

A summary of this simulation set is in the "Mis Ran 8" column of 6. The generating model is a random slope model, with the random slope on a cluster level covariate,

$$y_{ik} = 1 - 2x_{1k} + (2 + U_{2k})x_{2ik} + \epsilon_{ik} \quad U_{2k} \sim N(0, 0.8), \quad \epsilon_{ik} \sim N(0, 0.5), \tag{19}$$

where $x_{1k} \sim N(3, 9)$ and $x_{2ik} \sim N(1, 25)$. There are 300 population clusters, with a random uniform number of population units per population cluster between 50 and 100. The sample contains 35 clusters and 20 units per clusters. The sampling of clusters was proportional to the magnitude of the random effect U_{2k}. Sampling of individuals within clusters is proportional to an independently generated random variable assigned to each element[6]. There are two estimated equations in this simulation set. One matches the generated model, and one removes the random slope U_{2k} and adds a random intercept U_{0k},

$$y_{ik} = \beta_0 + U_{1k}x_{1k} + (\beta_2 + U_{2k})x_{2ik} + \epsilon_{ik}, \;\; U_{2k} \sim N(0, \sigma^2_{2k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon) \tag{20}$$

$$y_{ik} = \beta_0 + \beta_1 x_{1k} + \beta_2 x_{2ik} + U_{0k} + \epsilon_{ik}, \;\; U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon_{ik} \sim N(0, \sigma^2_\epsilon). \tag{21}$$

The sampling scheme is informative sampling for both estimated models.

5.7.1 Summary

The results from this simulation set are in Figure 7. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method matched well the estimation using the RHS method. There are no differences to highlight.

The coverage of the confidence intervals of RHS and PSHGR are mostly similar, with

[6]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5, 5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 20 are between 84% to 94%, and for the variance components they are between 28% and 89%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 20 are between 82% to 95%, and for the variance components they are between 51% and 93%. The number of confidence intervals for PSHGR covering the true parameter appears lager than the RHS intervals, especially for the σ_{2k}^2 parameter from the estimated model in Equation 20. This may indicate a problem with the variance estimator for PSHGR. To verify this, the coverage of the confidence intervals for the expected parameter value should be obtained. RHS was able to produce sandwich estimator variances for between 95% and 100% of the simulation runs, while PSHGR was able to produce design based estimator variances for 100% of the simulation runs. In addition, for the estimated model in Equation 21, there was one simulation for each of the the weighted scaled 2, unweighted and weighted scaled 1 estimates that did not converge for RHS after incrementing the number of quadrature points from 15 to 31.

Figure 7: Results for Misspecification of Random Variables, Simulation Set 8
Generated Model - Equation 19

The second estimated model contains model misspecification. The variance from the dropped random slop is split between the estimated variance of the random intercept and the estimated variance fo the random error, as is expected from the description in Section 8.1. The estimates of β were not affected by the model misspecification. The addition of the weights does not help compensate for this model misspecification.

Both estimated models contain informative sampling, the effects of which can be seen in the unweighted estimates of the β_{2ik}, σ^2_{2k} and σ^2_{0k} parameters. In the first estimated model, the unweighted estimate of β_{2ik} is larger than the weighted estimates, and the unweighted estimate of σ^2_{2k} is smaller than the weighted estimates due to oversampling larger values of U_{2k}. In the estimated model from Equation 21, the effect of the informative sampling on the β_{2ik} is the same as in Equation 20. In addition, the unweighted estimate of σ^2_{0k} is biased low, which can be seen when comparing it to the unweighted estimate of σ^2_{0k} from Equation 18 that does not contain the informative sampling.

All of the weighted estimates performed similarly for the β coefficients, however the variance for the weighted unscaled estimates is larger. The pattern in the variance components still holds, the weighted unscaled estimates are biased, the weighted scaled 1 estimates overcompensates for the bias and the weighted scaled 2 estimates are between the weighted unscaled and weighted scaled 1 estimates. The exception to this are the estimates of σ^2_ϵ for the estimated model in Equation 21, where the scaled 1 estimates provide more bias in the same direction as the weighted unscaled estimates. The weighted scaled 2 estimates are still between the unweighted and the weighted scaled 1 estimates. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.8 Misspecification of Stratification Layers - Stratified / Clustered Sampling - Simulation Set 9

A summary of this simulation set is in the "Mis Strat 9" column of Table 7. Let there be two strata where $I_{h==1}(I_{h==2})$ is an indicator variable that the element is in the first (second) stratum, respectively. Within each stratum, there is a layer of clustering. The generating model is a clustered/stratified model,

$$y_{ihk} = -3 + 8I_{h==1} + U_{01k}I_{h==1} + U_{02k}I_{h==2} + \epsilon_{ihk} \tag{22}$$

$$U_{01k} \sim N(0,1), U_{02k} \sim N(0,5), \epsilon_{ihk} \sim N(0,0.5), \text{Cov}(U_{01k}, U_{02k}) = 0.$$

This model allows the variance of the clusters in the first stratum to be different from the variance of the clusters in the second stratum. Within each of the two strata, there are 30 population clusters, with a random uniform number of population elements per population cluster between 50 and 100 units. The sample includes 5 clusters from each stratum, and 20 units from each cluster. Sampling of clusters within a stratum is proportional to an independently generated random variable assigned to each cluster[7]. Sampling of elements within a cluster is proportional to an independently generated random variable assigned to each element[8].

There are two estimated models in this simulation set. One matches the generated

[7]Each cluster was assigned a random variable $a_k \sim \text{Uniform}(-5,5)$. They were then sampled proportional to $(1+\exp(-a_k))^{-1}$.

[8]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5,5)$. They were then sampled proportional to $(1+\exp(-a_{ik}))^{-1}$.

model, and one removes the layer of stratification to estimate a cluster only scheme,

$$\begin{aligned} y_{ihk} &= \beta_0 + \beta_1 I_{h==1} + U_{01k} I_{h==1} + U_{02k} I_{h==2} + \epsilon_{ihk}, \qquad (23) \\ & U_{01k} \sim N(0, \sigma^2_{01}), U_{02k} \sim N(0, \sigma^2_{02}), \epsilon_{ihk} \sim N(0, \sigma^2_{\epsilon}), \text{Cov}(U_{01k}, U_{02k}) = \sigma^2_{01k.02k}, \\ y_{ihk} &= \beta_0 + U_{0k} + \epsilon_{ihk} \\ & U_{0k} \sim N(0, \sigma^2_{0k}), \epsilon_{ihk} \sim N(0, \sigma^2_{\epsilon}). \qquad (24) \end{aligned}$$

The sampling scheme is at random for both estimated models

These results are presented with the results of an additional simulation. This simulation used the same generating model, but uses informative sampling for the clusters. The generating model is

$$\begin{aligned} y_{ihk} &= -3 + 8 I_{h==1} + U_{01k} I_{h==1} + U_{02k} I_{h==2} + \epsilon_{ihk} \qquad (25) \\ & U_{01k} \sim N(0, 1), U_{02k} \sim N(0, 5), \epsilon_{ihk} \sim N(0, 0.5), \text{Cov}(U_{01k}, U_{02k}) = 0, \end{aligned}$$

and there was one estimating equation,

$$\begin{aligned} y_{ihk} &= \beta_0 + U_{0k} + \epsilon_{ihk} \qquad (26) \\ & U_{0k} \sim N(0, \sigma^2_{0k}), \epsilon_{ihk} \sim N(0, \sigma^2_{\epsilon}). \end{aligned}$$

Sampling of clusters within a stratum is proportional to the magnitude of the random effect, U_{01k} or U_{02k}, assigned to each cluster. Sampling of elements within a cluster is proportional to an independently generated random variable assigned to each element[9]. All the other components of the sampling scheme are the same as described after Equation 24.

The sampling scheme is informative sampling.

[9]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5, 5)$. They were then sampled proportional to $(1 + \exp(-a_{ik}))^{-1}$.

5.8.1 Results Summary

The results from this simulation set are in Figure 8. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. The differences that are visible in Figure 8 include all the estimates of $\sigma^2_{01k.02k}$ when the estimated model is in Equation 23. This difference is due to the very small point estimates (and no variance) in the PSHGR estimates. In the same estimated model, the PSHGR weighted scaled 1 estimates of σ^2_{02k} have a much larger 0.975 empirical quantile than RHS. In addition, the PSHGR weighted unscaled estimates of σ^2_{0k} from the estimated model in Equation 24 has a larger 0.025 and 0.975 empirical quantile than RHS. These and other differences not large enough to be seen in Figure 8 are in Section 8.1.

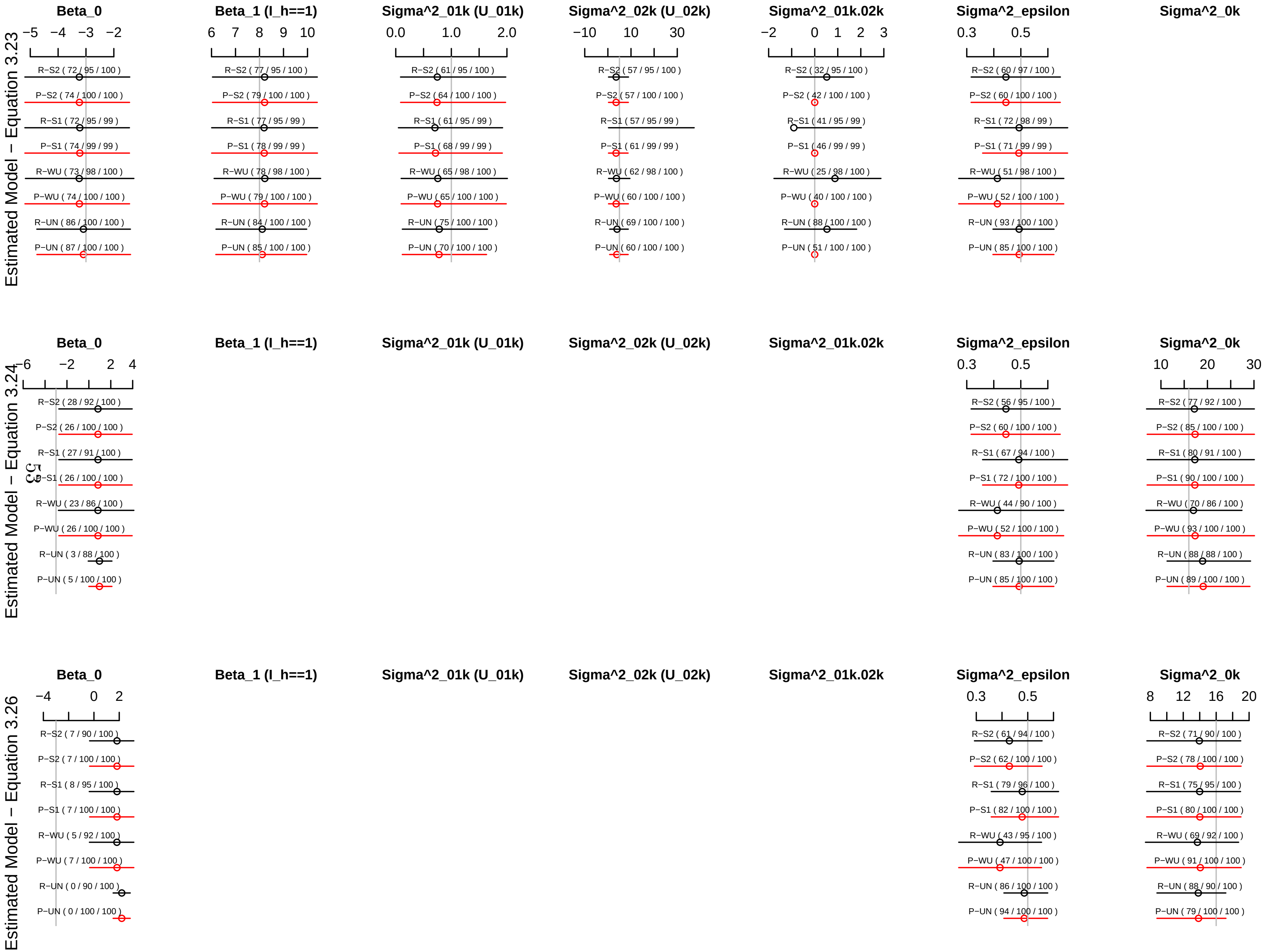


Figure 8: Results for Misspecification of Stratification Layers, Simulation Set 9
Generated Model - Equation 22

The coverage of the confidence intervals of RHS and PSHGR are similar (except for the $\sigma^2_{01k.02}$ intervals) with the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 23 are between 74% to 81%, and for the variance components they are between 52% and 93%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 20 are between 74% to 87%, and for the variance components they are between 52% and 85%. The confidence intervals for PSHGR do not capture the $\sigma^2_{01k.02k}$ well because many of the estimated variances of the point estimates were negative. RHS was able to produce sandwich estimator variances for between 86% and 100% of the simulation runs, while PSHGR was able to produce design based estimator variances for 100% of the simulation runs.

The second and third estimated models contain model misspecification as the stratified/clustered model was reduced to a clustered model. As expected, the estimated intercept became the average of the two strata intercepts (as the sample size had 50% from each stratum) and the estimate of the random intercept includes the variance of the means of the strata and the two random effects. The estimate of the random error did not change. For more description see Section 8.1. The third model also includes model misspecification and informative sampling. The addition of the weights does not help compensate for the model misspecification.

The third estimated model contains informative sampling. The unweighted estimate of β_0 exhibits bias from the informative sampling. This bias is reduced by the weighted estimates, but not eliminated. We also see the bias in the unweighted estimation of σ^2_{0k}. Note that the unweighted estimate from the estimated model in Equation 24 is larger than the weighted unscaled estimate from the same estimated model. However, the unweighted estimate of σ^2_{0k} from the esimtaed model in Equation 26 is smaller than the weighted unscaled estimate from the same simulation . We also see that all the means of the estimates of σ^2_{0k} from the estimated model in Equation 24 are larger than the true value, whereas for the same parameter in the estimated model from Equation 26 the means of the esimates are smaller than the true value. This shows again that the weighted estimates help compensate for the model misspecification, but do not eliminate it.

All the weighted estimates perform similarly for the β coefficients. The weighted estimates are all quite similar for the estimates of the variance components of the random slopes. They are closer together than the previous simulations estimates of the random error. The estimates of the variance components are exhibiting the same behavior as before, with the weighted unscaled as biased, the weighted scaled 1 overcompensating for the bias and the weighted scaled 2 between the weighted scaled 1 and the weighted unscaled estimates. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.9 Misspecification of the Stratification Layering - Clustered/Stratified Sampling - Simulation Set 10

A summary of this simulation set is in the "Mis Strat 10" column of Table 7. The sampling structure first samples clusters and within each cluster there are two strata. Let $I_{h==1}(I_{h==2})$ be an indicator variable that the element is in the first (second) stratum, respectively. The generating model is a random intercept model that takes into account the clustering and stratification,

$$\begin{aligned} y_{ikh} &= -3 + 8I_{h==1} + U_{0k} + \epsilon_{ikh} \\ & U_{0k} \sim N(0,5), \quad \epsilon_{ikh} \sim N(0,0.5), \end{aligned} \tag{27}$$

where the effect of being in a given stratum is the same regardless of cluster membership. There are 30 population clusters, each containing two strata. Each stratum contains a random uniform number of population elements per population cluster between 25 and 50. The sample includes 5 clusters. Within each of the 5 clusters, there are two strata, and 10 elements are sampled from each stratum. Sampling of clusters is proportional to an independently generated random variable assigned to each cluster[10]. Sampling of elements within a cluster is proportional to an independently generated random variable assigned to each element[11].

There are two estimated models in this simulation set. One matches the generated model, and one removes the layer of stratification to estimate a cluster only scheme,

$$y_{ikh} = \beta_0 + \beta_1 I_{h==1} + U_{0k} + \epsilon_{ikh}, \; U_{0k} \sim N(0,\sigma^2_{0k}), \;\; \epsilon_{ikh} \sim N(0,\sigma^2_{\epsilon}) \tag{28}$$

$$y_{ihk} = \beta_0 + U_{0k} + \epsilon_{ihk}, \;\; U_{0k} \sim N(0,\sigma^2_{0k}), \;\; \epsilon_{ihk} \sim N(0,\sigma^2_{\epsilon}). \tag{29}$$

[10]Each cluster was assigned a random variable $a_k \sim$ Uniform$(-5,5)$. They were then sampled proportional to $(1+\exp(-a_k))^{-1}$.

[11]Each element was assigned a random variable $a_{ik} \sim$ Uniform$(-5,5)$. They were then sampled proportional to $(1+\exp(-a_{ik}))^{-1}$.

Similar to the previous simulation set, these results are presented with the result of an additional simulation. This simulation used the same generating model, however the sampling scheme includes informative sampling for the clusters. The generating model is

$$\begin{aligned} y_{ikh} &= -3 + 8I_{h==1} + U_{0k} + \epsilon_{ikh} \\ &\quad U_{0k} \sim N(0,5), \quad \epsilon_{ikh} \sim N(0,0.5) \end{aligned} \tag{30}$$

and there was one estimating model,

$$\begin{aligned} y_{ihk} &= \beta_0 + U_{0k} + \epsilon_{ihk} \\ &\quad U_{0k} \sim N(0,\sigma^2_{0k}), \quad \epsilon_{ihk} \sim N(0,\sigma^2_{\epsilon}). \end{aligned} \tag{31}$$

Sampling of clusters is proportional ot the magnitude of the random effect, U_{0k}. Sampling of elements within a cluster is proportional to an independently generated random variable assigned to each element[12]. The case in which the estimating model matched the generating model was not run due to space considerations.

The sampling scheme is missing completely at random for the estimating models in Equations 28 and 29, and it is informative for the estimating model in Equation 31.

5.9.1 Summary

The results from this simulation set are in Figure 9. A detailed description of the results is in Section 8.1.

In this simulation set, the estimation using the PSHGR method mostly matched the estimation using the RHS method. There are some differences between the PSHGR and RHS estimates that are large enough to be seen in Figure 9. The PSHGR weighted scaled 1 estimate of σ^2_{ϵ} from the estimating model in Equation 29 has a much lower 0.025 quantile

[12]Each element was assigned a random variable $a_{ik} \sim \text{Uniform}(-5,5)$. They were then sampled proportional to $(1+\exp(-a_{ik}))^{-1}$.

and mean than the corresponding RHS estimate. The PSHGR unweighted estimate of σ^2_{0k} has a lower 0.025 quantile than the corresponding RHS estimate. The PSHGR weighted scaled 2 estimate of σ^2_{0k} has a lower 0.975 quantile and mean than the corresponding RHS estimate. Finally, the PSHGR scaled 1 estimate of σ^2_{ϵ} has a lower 0.025 quantile and mean than the corresponding RHS estimate.

The coverage of the confidence intervals of RHS and PSHGR are similar, with the RHS 95% confidence intervals of the β coefficients for the estimated model in Equation 28 is between 70% and 95%, and for the variance components the coverage is between 50% and 84%. The coverage of the PSHGR 95% confidence intervals of the β coefficients for the estimated model in Equation 28 is between 75% and 90%, and for the variance components the coverage is between 50% and 80%.

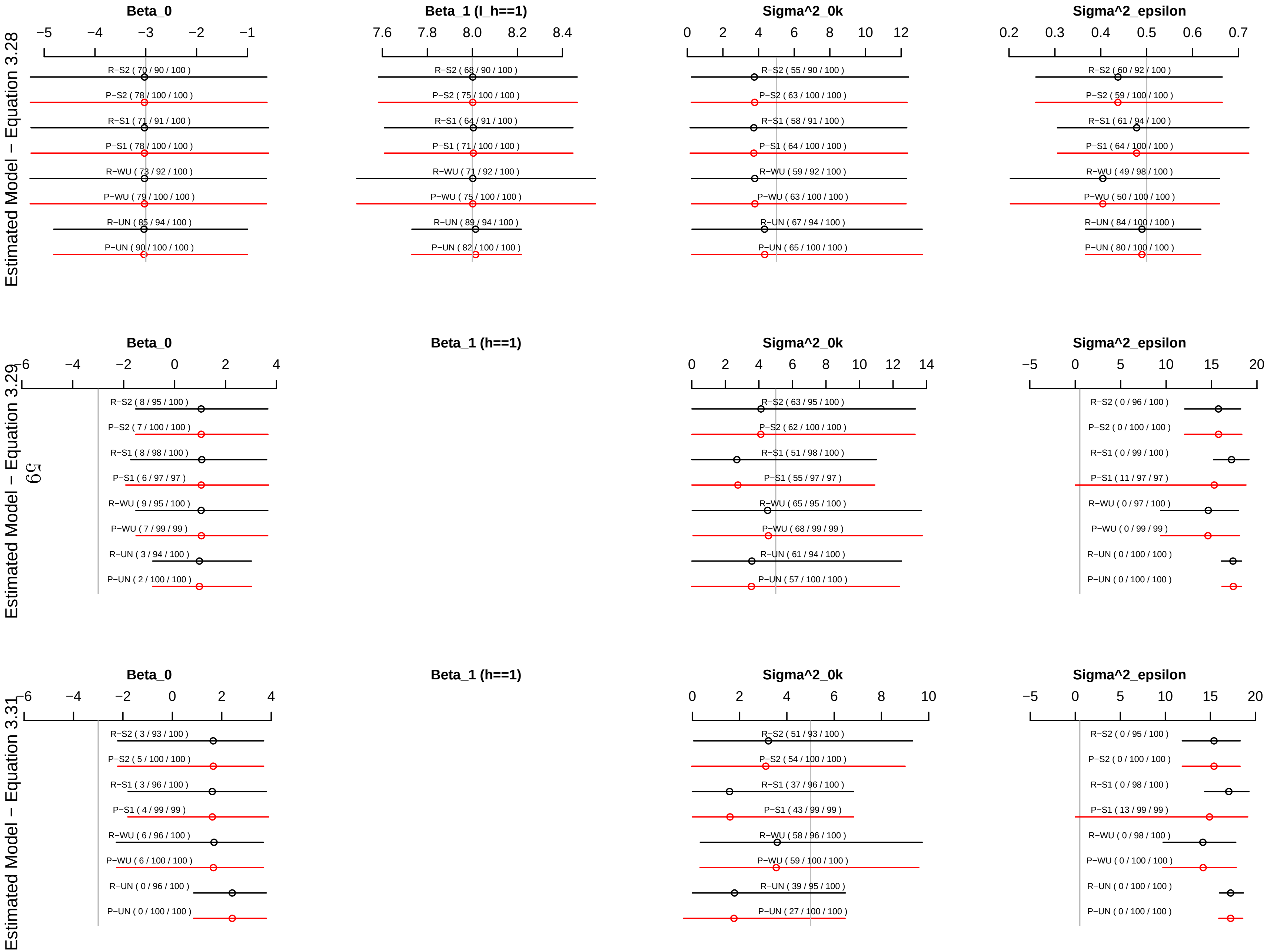


Figure 9: Results for Misspecification of Stratification Layers, Simulation Set 10
Generated Model - Equation 27

The second and third estimated models contains model misspecification. When the stratification layer was removed, it had the same effect as losing a fixed effects variable that varied according to cluster – the intercept estimate and the variance of the random error both changed. See Section 8.1 for more details. The addition of the weights does not help compensate for this model misspecification.

The third estimated model contained informative sampling, the effects of which can be seen in the unweighted estimates of the β_0, and σ^2_{0k} parameters. The all of the estimates (and especially the unweighted estimate) of σ^2_{0k} are smaller in the estimated model from Equation 31 than the corresponding estimates from the estimated model in Equation 29 The use of the weights helped to compensate for the informative sampling bias, but did not completely remove the bias.

All of the weighted estimates performed similarly for the β coefficients, however the variance for the weighted unscaled estimate is larger for the estimate of the stratification indicator. In addition, there are instabilities in the PSHGR estimation of σ^2_ϵ when using the scaled 1 weights. The pattern in the weighted estimates of the variance components is that the weighted scaled 1 has more bias in the same direction than the weighted unscaled estimate (except for the estimates of σ^2_ϵ when the estimated model is from Equation 28). The usual pattern is that the weighted scaled 1 estimates compensate (or overcompensate) for the weighted unscaled bias. Also unusual is the larger variance for the unweighted estimates of σ^2_{0k} in both the RHS and PSHGR estimates from the estimated model in Equation 28. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.10 Misspecification of Stratification Layers - Stratified/Clustered/Stratified Sampling - Simulation Set 11

A summary of this simulation set is in the "Mis Strat 11" column of Table 7. The sampling structure is a three stage stratify/cluster/stratify scheme where each layer of stratification has two strata. Let $I_{h1==1}(I_{h1==2})$ be an indicator that the element is in the first (second) top level strata, respectively. Let $I_{h2==1}(I_{h2==2})$ be an indicator that the element is in the first (second) lower level strata, respectively. The generating model is a random intercept model that takes into account the clustering and stratification,

$$\begin{aligned} y_{ih_1kh_2} &= 7 - 8I_{h_1==2} - 10I_{h_2==2} + U_{01k}I_{h_1==1} + U_{02k}I_{h_1==2} + \epsilon_{ih_1kh_2} \\ & U_{01k} \sim N(0,1), \quad U_{02k} \sim N(0,5), \quad \epsilon_{ih_1kh_2} \sim N(0,0.5). \end{aligned} \tag{32}$$

This generating model has separate means for the two top level strata where the effect of being in top level strata 1 is 5 and the effect of top level strata 2 is -3. The clusters within top level strata $h_1 == 1$ have a different random intercept variance than the clusters within top level strata $h_1 == 2$. Within each stratum / cluster, the effect of being in the bottom level second strata is the same regardless of cluster. The effect of being in lower level stratum 1 is 2 and the effect of being in lower level stratum 2 is -8. Thus the mean of a unit in the first top layer strata and the first lower level strata is 7 , the mean for the first top layer strata and the second lower level strata is -3, the mean for the second top level stratum and the first lower level stratum is -1, and the mean for the second top level stratum and the second lower level stratum is -11.

Each of the two upper level stratum contains 300 population clusters. Each cluster contains two lower level strata. Each lower level strata contains a random uniform number of population elements between 50 and 100. Within each top level strata, five clusters are sampled proportional to an independently generated random variable. Within each sampled cluster, 20 elements are sampled from each of the two strata. There are 400 elements in the sample.

There are four estimated models in this simulation set. The first estimating model matches the generating model. The second and third estimating models drop one (either the top or the bottom) layer of stratification. Finally the fourth estimating model drops both layers of stratification and has a cluster only model. The four estimated models are

$$
\begin{aligned}
y_{ijkl} &= \beta_0 + \beta_1 * (I_{ik} \in \text{s1= } 2) + \beta_2 * (I_{ijkl} \in \text{s2=2}) + U_{0k1} * (I_{ijkl} \in \text{s1=1}) \quad (33)\\
&+ U_{0k2} * (I_{ijkl} \in \text{s1=2}) + \epsilon_{ik}\\
&\quad U_{0k1} \sim N(0, \sigma^2_{0k1}), \quad U_{0k2} \sim N(0, \sigma^2_{0k2}), \quad \epsilon \sim N(0, \sigma^2_{\epsilon}),\\
y_{ik} &= \beta_0 + \beta_1 * (I_{ijkl} \in \text{S2=2}) + U_{0k} + \epsilon_{ik} \quad (34)\\
&\quad U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon \sim N(0, \sigma^2_{\epsilon}),\\
y_{ijkl} &= \beta_0 + \beta_1 * (I_{ik} \in \text{s1= } 2) + U_{0k1} * (I_{ijkl} \in \text{s1=1})\\
&+ U_{0k2} * (I_{ijkl} \in \text{s1=2}) + \epsilon_{ik} \quad (35)\\
&\quad U_{0k1} \sim N(0, \sigma^2_{0k1}), \quad U_{0k2} \sim N(0, \sigma^2_{0k2}), \quad \epsilon \sim N(0, \sigma^2_{\epsilon}),\\
y_{ik} &= \beta_0 + U_{0k} + \epsilon_{ik} \quad (36)\\
&\quad U_{0k} \sim N(0, \sigma^2_{0k}), \quad \epsilon \sim N(0, \sigma^2_{\epsilon}).
\end{aligned}
$$

This sampling scheme is sampling completely at random for all of the estimating models.

5.10.1 Summary

The results from this simulation set are in Figure 10. A detailed description of the results is in Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matches the estimation using the RHS method. However, there are many differences between the PSHGR and RHs estimates large enough to be seen in Figure 10. First consider the estimating model in Equation 33. The PSGHR weighted scaled 1 estimates of σ^2_{01k} have a smaller mean and a smaller 0.975 quantile than the corresponding RHS estimates. The estimates of σ^2_{02k} and $\sigma^2_{01k.02k}$ have obvious differences. For the estimating model in Equation 34, the PSHGR unweighted estimate of σ^2_{0k} has a larger 0.025 and 0.975 quantiles and mean than the corresponding RHS estimate. The mean of the PSHGR weighted unscaled estimates of σ^2_{0k} has a larger mean than the corresponding RHS estimates. The mean of the PSHGR weighted scaled 2 estimates of σ^2_{0k} is larger than the associated RHS estimates. There are many differences from the estimated model from Equation 35. The mean of the PSHGR weighted unscaled estimates of β_0 is smaller than the corresponding RHS estimate. The PSHGR weighted scaled 1 0.025 quantile for β_0 is smaller than the corresponding RHS estimate. The PSHGR mean and 0.025 quantile for the weighted unscaled estimates of β_1 are larger than the corresponding RHS estimates. The PSHGR mean and 0.975 quantiles of the weighted scaled 1 estimates of β_1 are larger than the corresponding RHS estimates.

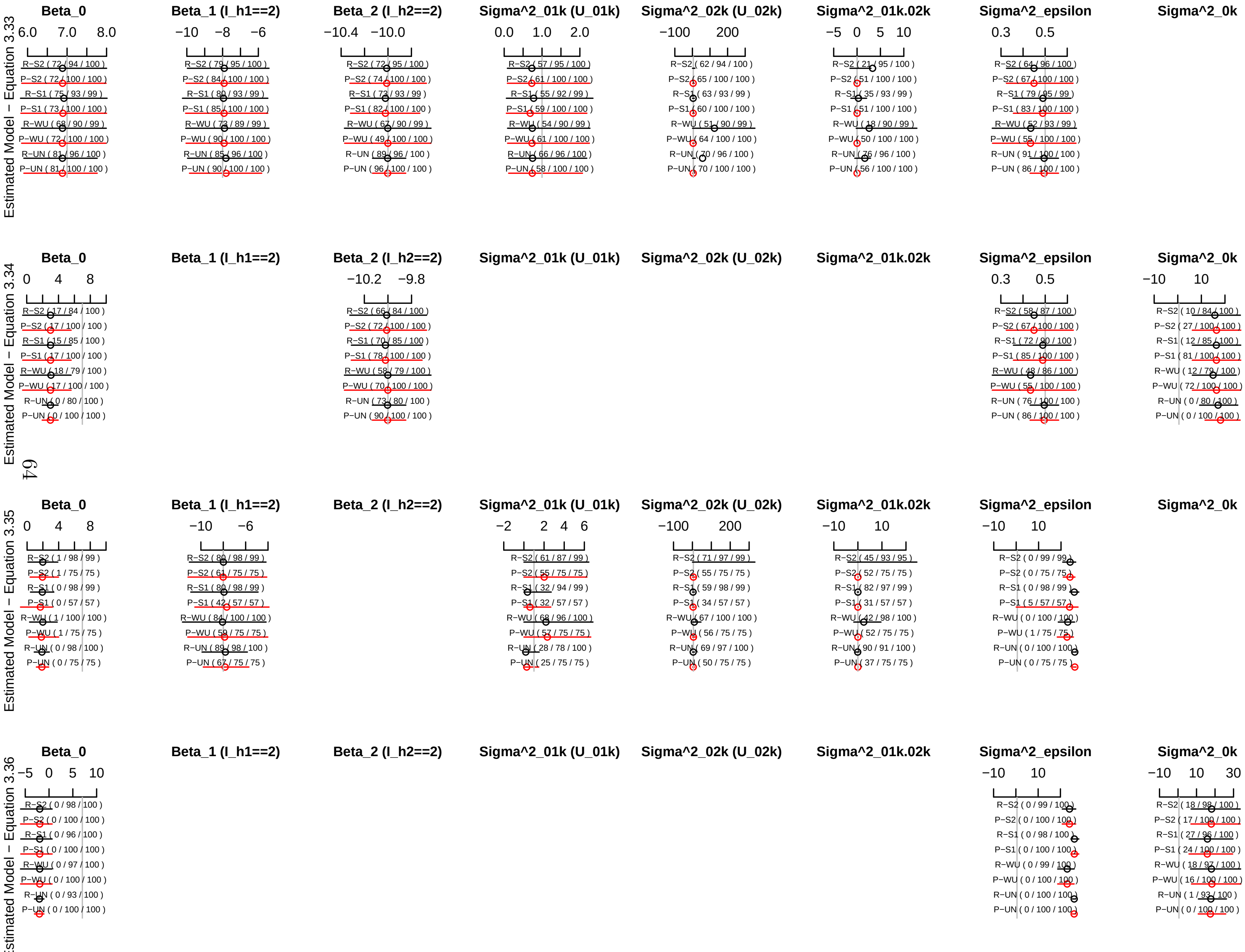


Figure 10: Results for Misspecification of Stratification Layers, Simulation Set 11
Generated Model - Equation 32

There is a large outlier in the RHS weighted scaled 2 estimates of σ^2_{01k} and σ^2_{02k} resulting in the mean of the esimates to be off of the scale of the graph. The RHS weighted unscaled estimates of $\sigma^2_{01k.02k}$have a much wider range and larger mean than the associated PSHGR estimates. Finally, the PSHGR weighted scaled 1 estimates of σ^2_ϵ have a lower 0.025 quantile and mean than the associated RHS estimates. For more details, see Section 8.1. The coverage of the confidence intervals of RHS and PSHGR are similar, with the RHS 95% confidence intervals for the β coefficients from the estimated model in Equation 33 are between 74% to 93% and for the variance components (not including $\sigma^2_{01k.02k}$) they are between 56% and 91%. The coverage of the PSHGR 95% confidence intervals for the β coefficients from the estimated model in Equation 33 are between 72% to 96% and for the variance components (not including $\sigma^2_{01k.02k}$) they are between 55% and 86%. RHS was able to produce sandwich estimator variances for between 75% and 100% of the simulation runs, while PSHGR was able to produce design-based estimator variances for between 57% and 100%.

The second, third and fourth rows of Figure 10 contain model misspecification involving dropping the top level, bottom level or both levels of stratification. The means of parameters are what is expected as described in Section 8.1. The misspecification is seen mostly in the estimates of β_0 and σ^2_ϵ. The addition of the weights does not compensate for this model misspecification.

There is no informative sampling in this simulation so there is no informative sampling bias.

The patterns in the different weightings are hard to see in this simulation due to the outlying observations. For the β coefficients, it appears that the weighted unscaled estimates has a larger variance, especially for the estimating model in Equation 34 and 35. For the variance estimates, most appear to follow the pattern that the weighted unscaled estimates are biased, the weighted scaled 1 (over) compensates for the bias and the weighted scaled 2 estimates are between the weighted unscaled and the weighted scaled 1 estimates. There are two parameters where the weighted scaled 1 apears to add bias in the same direction of the weighted unscaled estimates, specifically the estimates of σ_ϵ^2 from the estimated models in Equations 35 and 36. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

5.11 Misspecification of Clustering Layers – Simulation Set 12

A summary of this simulation set is in the "Mis Clust 12" column of Table 7. The sampling structure first clusters on the top layer clusters (denoted k_1), then selects lower level clusters (denoted k_2) within the top layer clusters. The generating model is a two-level random slope model to fit the cluster/cluster design,

$$\begin{aligned} y_{ik_1k_2} &= 5 + U_{0k_1} + U_{0k_1k_2} + \epsilon_{ik_1k_2} \\ & \quad U_{0k_1} \sim N(0,5), \quad U_{0k_1k_2} \sim N(0,1), \quad \epsilon_{ik_1k_2} \sim N(0,0.5). \end{aligned} \tag{37}$$

There are 30 top level population clusters and within each top level population cluster there are 10 bottom level population clusters with a random uniform number of population units per cluster between 25 and 50. The sample contains 5 top level clusters, 5 bottom level clusters and 3 elements per bottom level cluster. The top level clusters are sampled proportional to first independent random variable, the bottom level clusters are sampled proportional to a second independent random variable, and the elements within the bottom cluster are sampled proportional to a third independently generated random variable. There are two estimating models in this simulations set, the first removes the bottom layer of clustering,

$$\begin{aligned} y_{ik_1k_2} &= 5 + U_{0k_1} + \epsilon_{ik_1k_2} \\ & \quad U_{0k_1} \sim N(0,\sigma^2_{0k1}), \quad \epsilon_{ik_1k_2} \sim N(0,\sigma^2_{\epsilon}), \end{aligned} \tag{38}$$

and the second removes the top layer of clustering,

$$\begin{aligned} y_{ik_1k_2} &= 5 + U_{0k_1k_2} + \epsilon_{ik_1k_2} \\ & \quad U_{0k_1k_2} \sim N(0,\sigma^2_{0k1k2}), \quad \epsilon_{ik_1k_2} \sim N(0,\sigma^2_{\epsilon}). \end{aligned} \tag{39}$$

Due to time constraints, none of the estimated models match the generating model. This sampling scheme is sampling completely at random for both estimated models.

5.11.1 Summary

The results from this simulation set are in Figure 11. For a complete description of the simulation results, see Section 8.1.

In this simulation, the estimation using the PSHGR method mostly matched the estimation using the RHS method. THere are some differences between the PSHGR and RHS estimates large enough to be seen in Figure 11. From the estimated model in Equation 38, the PSHGR empirical confidence intervals for the weighted scaled 1 and weighted scaled 2 estimates of σ^2_{0k1} are longer than the corresponding RHS intervals. The 0.025 quantile of the PSHGR weighted scaled 1 and weighted scaled 2 estimates of σ^2_{ϵ} are smaller than the corresponding RHS quantiles. For the estimated model in Equation 39, the 0.975 quantiles for the weighted unscaled, weighted scaled 1 and weighted scaled 2 are larger for the RHS intervals than for the corresponding PSHGR intervals.

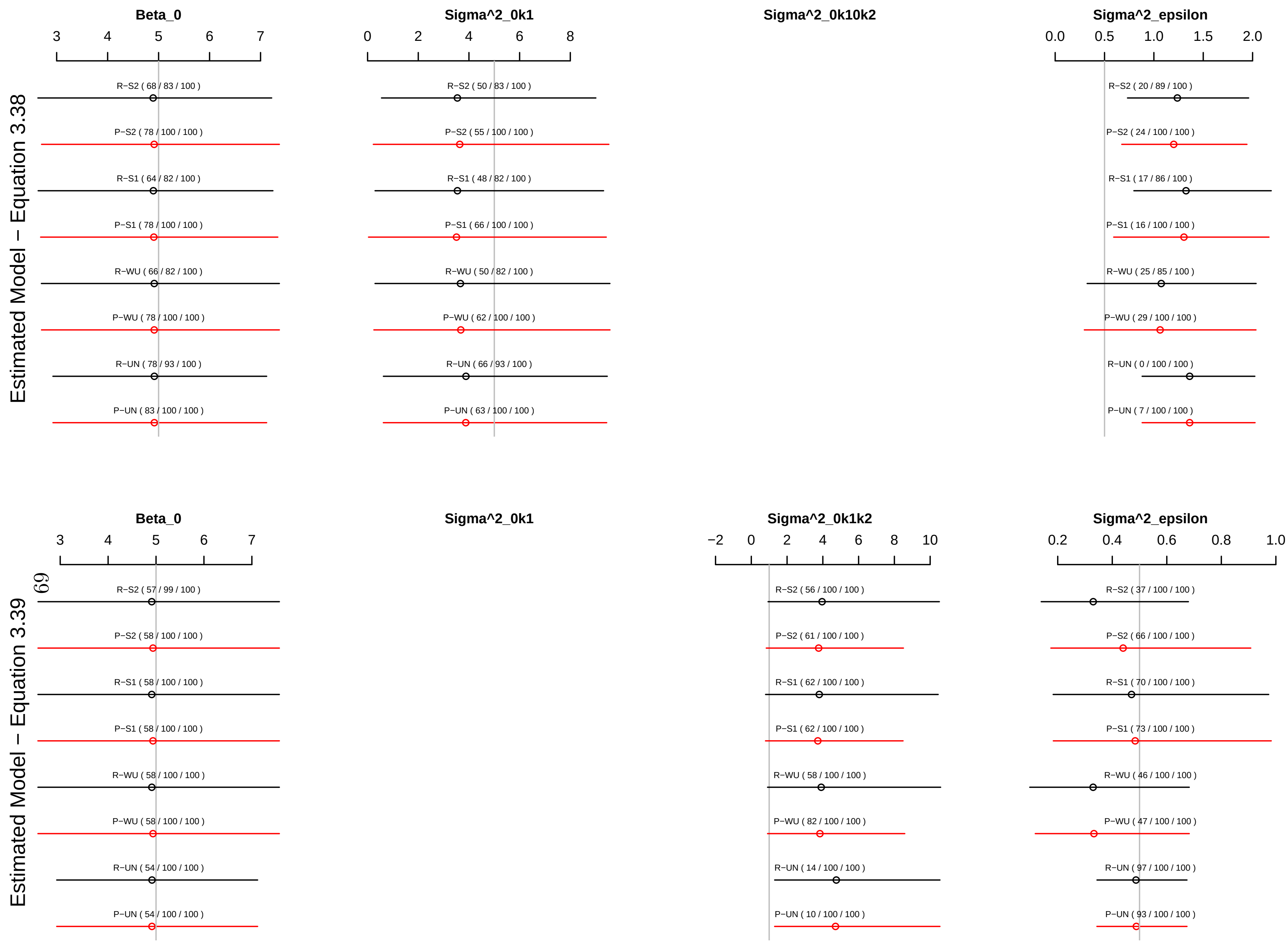


Figure 11: Results for Misspecification of Clustering Layers, Simulation Set 12
Generated Model - Equation 37

For the PSHGR 0.025 quantile of the weighted unscaled estimate of σ_ϵ^2 is larger than the corresponding RHS quantile. The mean of the PSHGR scaled 2 estimates of σ_ϵ^2 is larger than the corresponding RHS mean. Finally, the 0.025 and 0.975 quantiles and the mean of the PSHGR weighted scaled 2 estimates of σ_ϵ^2 are larger than the corresponding RHS estimates. For more details, see Section 39.

The coverage of the confidence intervals of RHS and PSHGR is not analyzed in this simulation as both estimated equations contain model misspecification.

The first and second rows both contain model misspecification, as the generating model is a three level random intercept model and the two estimating models are two level random intercept models. In these simulations, the variance from the cluster level that was dropped was merged into the remaining cluster level or the random error term. For a description of the expected results, see Section 8.1. The addition of the weights did not compensate for the model misspecification.

There is no informative sampling in this simulation so there is no informative sampling bias.

The means of all weighted estimates are similar for the β coefficient. The variance of the unweighted estimates is smaller than for the weighted estimates. For the σ_ϵ^2 from the estimated model in Equation 39, we see the pattern where the weighted unscaled estimates are biased, the scaled 1 estimates (over) compensate for the bias and the scaled 2 estimates are between the weighted unscaled and the weighted scaled 1 estimates. However, for the estimates of the variance components from the estimated model in Equation 38, we see that the scaled 1 weights are adding bias in the same direction as the weighted unscaled weights. With the differences in estimates of $\sigma_{0k2.0k2}^2$, the pattern is difficult to determine. The unweighted estimates a smaller 0.975, 0.025 quantile spread than the weighted estimates in all these simulations. When the spreads of the weighted estimates vary, then the weighted unscaled spread is the largest, followed by the weighted scaled 2 estimates spread and the weighted scaled 1 estimates spread.

6 Mean Squared Error Comparisons of the Simulations

It is not clear how to compare the different methodologies (PSHGR vs. RHS) crossed by the different weightings. A criterion such as AIC or BIC is desired, however it is not clear if these are appropriate. AIC and BIC aid in model selection, however the insertion of the sampling weights in different places doesn't necessarily fall into model selection. To help find a good metric, I propose two different calculations based on the mean squared error. I evaluate the simulations based on their metrics and discuss the strengths and weaknesses. I do not believe these are good metrics to evaluate the simulations, but they identify issues that need to be considered when determining a metric.

Relative Square Root Mean Squared Error (RRMSE)

Let $\hat{\beta}_1$ be the estimate of β_1 and let n be the number of simulation runs that produced point estimates. Then $RRMSE = \sqrt{\sum_{i=1}^{n} n^{-1}\beta_1^{-2}(\hat{\beta}_1 - \beta_1)^2}$. This is the square root of the mean squared error that is scaled by the magnitude of the parameter. This metric balances the bias and the variance for each parameter.

RRMSE is a measure of the model misspecification and informative sampling. Often, the model misspecification dominates the $RRMSE$. To help compensate for this, the $ARRMSE$ is also computed.

Adjusted Relative Square Root Mean Squared Error (ARRMSE)

Similar to the $RRMSE$, however instead of using the true value of the parameter we use the anticipated value of the parameter value given the model misspecification. For example, if β_{1A} is the anticipated value of the parameter given the model misspecification, then $ARRMSE = \sqrt{\sum_{i=1}^{n} n^{-1}\beta_{1A}^{-2}(\hat{\beta}_1 - \beta_{1A})^2}$ where n is the number of simulation runs out of 100 that produced point estimates. This $ARRMSE$ removes the model misspecification component from the $RRMSE$, and measures the effects of informative sampling.

For more information on the derivation of the anticipated parameter values, see the description for the simulation in question in Section 8.1. The anticipated parameter values are tabulated in Section 8.5.

The values of the $RRMSE$ and $ARRMSE$ for each estimate in the simulations are in Section 8.5. To summarize this data, I added the $RRMSE$ ($ARRMSE$) values of each estimate for a given estimating model, methodology (PSHGR vs. RHS) and weighting scheme. This has advantages and disadvantages. The advantage is that when a model is estimated, the estimates of the parameters that are used must come from one estimating set. For example, I can not choose an estimating model and the estimate the fixed effects using, for example, PSHGR unweighted estimates and then estimate the random effects using RHS weighted scaled 2 estimates. This merges all estimates from a given estimated model together within one framework. The problem is that when the scales differ and when there is model misspecification, the estimate of one parameter in the model can dominate the mean squared error calculation. For that reason, the relative MSE is used (i.e. diving by the true/anticipated value) and both $RRMSE$ and $ARRMSE$ are presented. However, when the true (or anticipated value) is zero, then the $RRMSE$ (or $ARMSE$) can not be computed.

Table 8 contains a summary of the results of the 12 simulation sets. The first column contains the name of the simulation set. The second column contains the equation number of the estimated equation. The $RRMSE$ for all the parameters of a given type of weight and method are then added together. In the subsequent columns, P and R in the given column represent the PSHGR and RHS weighting scheme that produces the smallest $RRMSE$, and PA and RA represent the smallest $ARRMSE$. Note that for estimated models in Equations 12, 15, 18 and 21 a random intercept is included in the estimated model instead of the random slope. Because the true parameter value of the random intercept variance is zero, the $RRMSE$ can not be computed however the $ARRMSE$ is computed and recorded. For the estimated models in Equations 23, 33 and 35, the true parameter value of $\sigma^2_{01k.02k}$ is zero. For these equations, the $RRMSE$ and $ARRMSE$ are computed without a contribution from the estimates of $\sigma^2_{01k.02k}$. More detailed summary tables are in Section 8.4

For a detailed description of the MSE results, see Section 8.4. The same weighting method generally produced the lowest MSE for both the PSHGR and RHS methodologies. When this is not the case (see table 8 for Equation numbers 4 and 31) it is due to differences in the methods described in Section 8.1.

The unweighted estimates generally provided the lowest $ARRMSE$. The cases where this is not true (see table 8 for Equation numbers 8 and 31) are due to informative sampling. The $AARMSE$ prefers the unweighted estimates due to their smaller variance. The bias induced by the informative sampling in these simulations is not large enough to penalize the unweighted estimates. The $RRMSE$ is more sensitive to model misspecification and appears to prefer the unweighted and weighted unscaled estimates. The preference for the weighted unscaled estimates occurs because these estimates show the most bias in the variance components. When the model is misspecified and the anticipated value of the variance component gets large causing a very large bias when compared to the true value of the parameter. That variance component dominates the sum of the $RRMSE$ and it is often the weighted unscaled estimates that are closest to the true value of the parameter. Ffor example, see the estimate of σ^2_ϵ in Figure 2 from the estimated model in Equation 5.

As described in Section 8.4, the level of informativeness is a big factor as to which type of weighting scheme is preferred.

	Eqn. Num.	Weighting Scheme with Lowest MSE			
		Unweighted	Weighted Unscaled	Weighted Scaled 1	Weighted Scaled 2
Mis Fix 1	3		P R		
	4	PA RA	R	P	
	5	PA RA	P R		
Mis Fix 4	7	P R			
	8	P R		PA RA	
	9	P R PA RA			
Mis Ran 5	11	P R			
	12	PA RA			
Mis Ran 6	14				P R
	15	PA RA			
Mis Ran 7	17	P R			
	18	PA RA			
Mis Ran 8	20			P R	
	21	PA RA			
Mis Strat 9	23	P R			
	24	PA RA			
	26	PA RA			
Mis Strat 10	28	P R			
	29	PA RA			
	31			RA	PA
Mis Strat 11	33	P		R	
	34	PA RA			
	35	PA RA	P R		
	36	PA RA			
Mis Clust 12	38	PA RA	P R		
	39		PA RA P R		

Table 8: Mean Squared Errors for each Simulation Set

7 Simulation Result Summary

This chapter provides new contributions or supports existing claims on each of five goals. This is accomplished through 12 sets of simulations that compare estimation methods (PSHGR or RHS) and scaling of weights (unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2) on correctly and incorrectly specified models, both with and without informative sampling.

The first goal is to compare the results from the different methods of inserting weights into LME models. This chapter compares the method of Rabe-Hesketh and Skrondal (2006) , which is the same as Asparouhov (2006), to the method of Pfeffermann et al. (1998). These simulations found that the RHS and PSHGR methods provide remarkably similar results. When the results are not similar, it is mostly due to sensitivities of the numerical quadrature to the number of quadrature points in the `gllamm()` function that implements the RHS method. Neither RHS nor PSHGR provided this direct comparison in their papers.

The second goal is to compare the sandwich estimator (used by RHS) and the design-based estimator (used by PSHGR) when obtaining the variances of the point estimates. When there is no model misspecification, the confidence intervals based on the sandwich estimator have similar coverage levels as the confidence intervals based on the design-based estimates. However, when there is model misspecification, the design-based confidence intervals have coverage that is unexpectedly large, implying that the variance estimates are too large. Neither RHS nor PSHGR provided a comparison in their papers and neither of them looks at the performance of the variance estimators in the presence of model misspecification.

The third goal of this chapter is to investigate the assertion that adding sampling weights can compensate for model misspecification in LME models. The simulations in this chapter indicate that the weights can help for model misspecification only when the model misspecification induces informative sampling. Bias related to a misspecified model that does not relate to the sampling design is unaffected by the sampling weights. Previous simulation studies did not study model misspecification.

The fourth goal of this chapter is to investigate the assertion that adding sampling weights can compensate for informative sampling in LME models. The simulations in this chapter support those conclusions. The inverse probability sampling weights can help compensate for bias induced by informative sampling, though they do not eliminate the bias. This supports the conclusions in the previous simulation papers.

The final goal of this chapter is to investigate the different scalings of the weights used in RHS, PSHGR, and ASP. These simulations found that the unweighted estimates have the smallest variance. However, when there is informative sampling, the unweighted estimates are biased. The weighted unscaled estimate corrects the bias in the fixed effects, but produces more bias in the random effects. The weighted scaled 1 estimates remove the bias in the fixed effects, and correct (or overcorrect) for the weighted unscaled bias in the random effects. The weighted scaled 2 estimates remove the bias in the fixed effects and have a bias between the weighted unscaled and weighted scaled 1 estimates in the random effects. There are times when the scaled 1 estimates have more bias in the same direction as the weighted unscaled estimates. The conditions upon which this occurs need to be further investigated. RHS, PSHGR and ASP tentatively recommended the weighted scaled 2 estimates. These simulations provide a good characterization of the relationship between the scaled estimates and demonstrate that the variance of the scaled 1 estimates is sometimes lower than the scaled 2 estimates.

Comparison of the weighting schemes over the different estimated models is difficult. To gain insight into the comparison, I computed the $RRMSE$ and $ARRMSE$ metrics and looked at their strengths and weaknesses. The $RRMSE$ metric incorporates informative sampling, model misspecification and variance and generally prefers the unweighted or weighted unscaled estimates. This is due to the low variance of the unweighted estimates and the pattern of bias in the weighted unscaled estimates. The $ARRMSE$ metric incorporates informative sampling and variance, and generally prefers the unweighted or weighted scaled 1 estimates. This is due to the low vairance of the unweighted estimates, and the slightly higher variance, but lower bias of the weighted scaled 1 estimates. None of the previous simulation papers attempted a metric across all estimates in a model.

This chapter contributes a new way to view the simulation results. RHS, ASP, KG and PSHGR produced tables of numbers that are difficult to read and make quick comparisons. The stacked line interval format of the displays in this chapter provides a quick visual way to compare all methods together and across multiple simulations.

The results of this chapter can be generalized to more complex LME models. This chapter addressed the effects of model misspecification and informative sampling on fixed effects in two scenarios; 1) biases confined to one one level (by removal of either the x_{1k} or x_{2ik} fixed variables in simulation sets 1 and 4, for example) and 2) biases spread across levels (by the removal of the random slope on x_{2ik} in simulation set s 7 and 8, for example). These scenarios can be easily generalized into more complex models. As the random effect structure increases and becomes more complex, I speculate that the bias in the random effects will become worse. This is because the ML estimates of the random effects are biased where the bias of one variance component depends on other variance components, as seen in the case of the estimates of σ^2_{0k} in simulation set 4 with the estimated model in Equation 9. The use of the weights adds to the bias of the random effects. Once one random effect estimate is biased, that bias may be propagated through to other variance components.

8 Appendix

8.1 Description of Simulation Results

8.1.1 Result Description for Misspecification of Fixed Variables - Simulation Set 1

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold.

Figure 12 contains a plot of the weighted scaled 1 estimates for β_2, the unweighted and weighted unscaled estimates of σ^2_{0k} and the unweighted estimates of σ^2_ϵ from the estimated model in Equation 4. The solid black lines are the upper and lower thresholds. From the figure, we see that there is one point that is outside the lines for the estimate of β_2, 11 and 6 points outside the lines for the unweighted and weighted unscaled estimates of σ^2_{0k} respectively, and one point outside the line for the estimate of σ^2_ϵ. The differences between the weighted scaled 1 estimates for β_2 are too small to be seen in Figure 2, however the differences in unweighted and weighted unscaled estimates for σ^2_{0k} can be seen as the means do not match each other. The differences in the unweighted estimates of σ^2_ϵ can also be seen in Figure 2. It appears that the means may be different for the weighted scaled 2 estimate of σ^2_{0k}, however all the individual estimation differences are less than the threshold.

For the estimates from the estimating model in Equation 5, note that the PSHGR and RHS estimates using the scaled 1 weights do not have 100 estimates. For PSHGR, simulation runs 21 and 94 did not converge in 500 iterations. For RHS, simulation runs

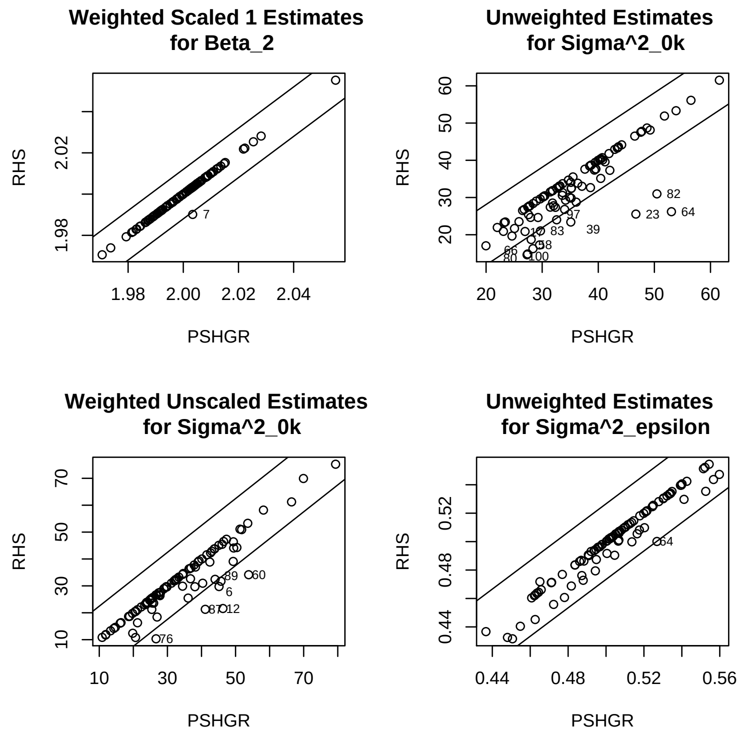


Figure 12: Comparison of PSHGR vs. RHS for Estimates from Equation 4

28 and 41 did not converge when the number of quadrature points were increased from 15 until 30. Figure 13 contains the unweighted estimates of σ^2_{0k} and the weighted scaled 1 estimate of σ^2_ϵ. For the estimates of σ^2_{0k}, there are a number of PSHGR estimates that range from 0 to 2 while the RHS estimates are all about 0.25. I believe that this is a problem with the RHS estimation, however his pattern should be looked into further. For the estimate of σ^2_ϵ the PSHGR weighted scaled 1 estimate of σ^2_ϵ for simulation run 90 is 345. I believe this is an instability with the PSHGR estimation and should be looked into further. The differences in the PSHGR and RHS estimates of σ^2_{0k} can not be seen in Figure 2, however the estimates of σ^2_ϵ appear to have different means in the figure.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 2 contains the summary of the estimated model from

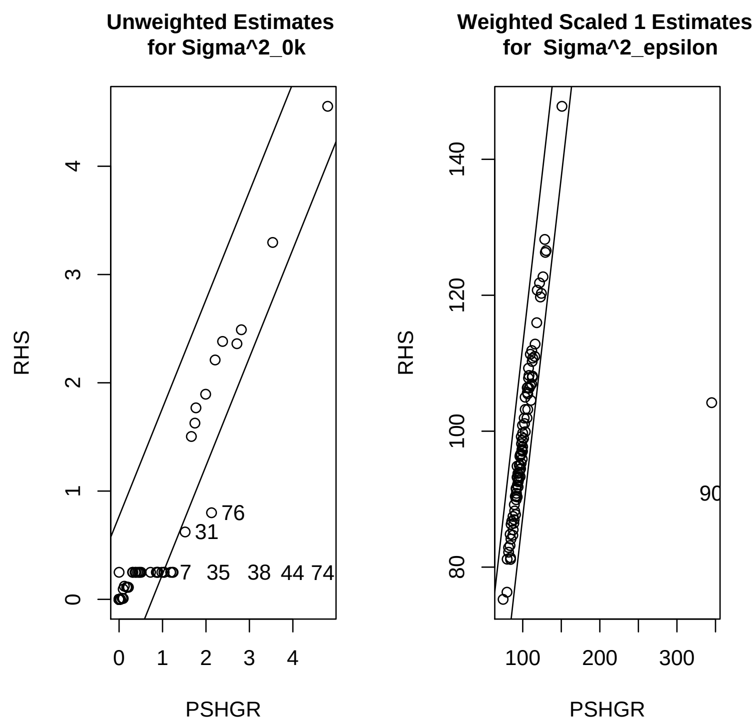


Figure 13: Comparison of PSHGR vs. RHS for Estimates from Equation 5

Equation 3, where the estimated model matches the generating model. We know that the unweighted estimates of β should be unbiased based on Section 6.2 of Searle et al. (1992). All of the estimation methods have minimal bias and comparable quantiles for the β parameters. It is well documented that the variance components are not necessarily unbiased. Specifically, the σ^2_{0k} parameter depends on the intra-class correlation. The intra-class correlation in this data set was $\frac{0.2}{0.2+0.5} = 0.29$, and the simulation results show a slight positive bias for the weighted estimate using unscaled weights. For weighted estimates using scaled 1 and scaled 2 weights, the biases are negative and approximately the same magnitude. The σ^2_ϵ parameter estimates have the following trends: the weighted unscaled estimates having larger negative bias, the weighted scaled 1 and weighted scaled 2 estimates having smaller positive bias.

In the middle panel, the estimated model from Equation 4 no longer contains the x_{1k} variable. The mean of the missing $-2x_{1k}$ term is -6, resulting in a new intercept estimate of 1-6=-5. In addition, the variance of the missing $-2x_{1k}$ term is 36, resulting in a new σ^2_{0k} estimate of approximately 36+0.5=36.5. The bias in σ^2_{0k} follows the same trend as first row of the simulation results. The bias in σ^2_ϵ is larger than from the model in Equation 3. As expected, the weighting does not do anything to help in this model misspecification.

In the bottom panel, the estimated model from Equation 5 no longer contains the x_{ik} variable. The mean of the missing $2x_{2ik}$ term is 2, resulting in a new intercept estimate of approximately 3. The variance of the missing $2x_{2ik}$ term is 100, resulting in a new σ^2_ϵ estimate of approximately 100.5. The estimate of σ^2_{0k} is more difficult to predict, as the unweighted and weighted scaled 1 estimates of σ^2_{0k} are occasionally negative. For the unweighted estimates, based on the calculations in §3.5 of Searle et al. (1992), $E(\hat{\sigma}^2_{0k}|\hat{\sigma}^2_{0k} \geq 0)$ is computed as the average of the 39 non-negative estimates, which is 0.79. We can compute $p = \Pr(\hat{\sigma}^2_{0k} < 0) \approx 0.61$ by assuming that this is a balanced simulation with the number of clusters as 35, the number of elements per cluster as 20, $\sigma^2_{0k} = 0.2$ and $\sigma^2_\epsilon = 100.5$. As a result $E(\hat{\sigma}^2_{0k}) = (1-p)E(\hat{\sigma}^2_{0k}|\hat{\sigma^2_{0k}} \geq 0) \approx 0.39 * 0.79 = 0.31$. Thus our theoretic estimate σ^2_{0k} is 0.31. Compare this to our actual results by allowing all of the negative $\hat{\sigma}^2_{0k} = 0$. We get an estimate of σ^2_{0k} over the 100 iterations of 0.31. Thus the simulated result for $\hat{\sigma}^2_{0k}$ matches the theoretical result. The scaled 1 case is computed similarly, but with 67 of the 100 iterations producing negative estimates of σ^2_{0k}. $E(\hat{\sigma}^2_{0k}|\hat{\sigma^2_{0k}} \geq 0)$ is computed as the average of the 33 non-negative estimates, which is 1.63. We can compute $p = \Pr(\hat{\sigma^2_{0k}} < 0) \approx 0.67$ by assuming that this is a balanced simulation with the number of clusters as 35, the number of elements per cluster as 20, $\sigma^2_{0k} = 0.2$ and $\sigma^2_\epsilon = 100.5$. As a result $E(\hat{\sigma}^2_{0k}) = (1-p)E(\hat{\sigma}^2_{0k}|\hat{\sigma^2_{0k}} \geq 0) \approx 0.33 * 1.63 = 0.54$. Thus our theoretical estimate of σ^2_{0k} is 0.54. Compare this to our actual results by allowing all of the negative $\hat{\sigma}^2_{0k} = 0$. We get an estimate of σ^2_{0k} over the 100 iterations of 0.52. It is assumed that the difference between 0.54 and 0.52 is due to the fact that this is not a balanced simulation. The mean scaled 2 estimate for RHS is 7.6, which has a bias of 7.1. There is an unexplained bias of 7.1-5.8=1.3, which I assume is attributed to the non-balanced nature of this simulation. In the weighted unscaled case, assume that the number of population elements per cluster is 75 (it is between 50 and 100), the number of sampled elements is 20 and $\sigma_\epsilon^2 = 100.5$. Then the bias bounds are approximately -5 to 94. The bias from the simulations is approximately 10, so the bias is within what is expected. Note that the ICC is now fairly small (0.2/100.7=0.002). As expected, the weighted estimates did not appear to compensate for the model misspecification.

8.1.2 Result Description for Misspecification of Fixed Variables – Simulation Set 4

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. In this set of simulations, there are a number of datasets that were problematic for all weighting schemes and all parameters. For example, when the estimating model is in Equation 7, the difference between the PSHGR and RHS estimates is greater than the threshold for all estimates for the data from simulation run 18. The plots to show these differences for each parameter and each scaling are not shown to conserve space.

When the estimating model is from Equation 8, the only parameter that produces differences between the PSHGR and RHS estimates that are greater than one threshold is σ^2_{0k}, as shown in Figure 14. For this parameter, for the unweighted estimates simulation run 80 is larger than the threshold, for the weighted unscaled estimates simulation runs 15 and 79 are larger than the threshold and for the weighted scaled 2 estimates simulation runs 36 and 63 are larger than the threshold.

When the estimating model is from Equation 9, the simulation runs 1, 84 and 96 produced differences between PSHGR and RHS larger than the threshold in many the parameter estimates. The plots to show these differences for each parameter and each scaling are not shown to conserve space. However, there were some notable differences in PSHGR and RHS in the unweighted and weighted scaled 1 estimates of σ^2_{0k}, as seen in Figure 15. Similar to what was seen in Figure 13, the PSHGR estimates appear to vary

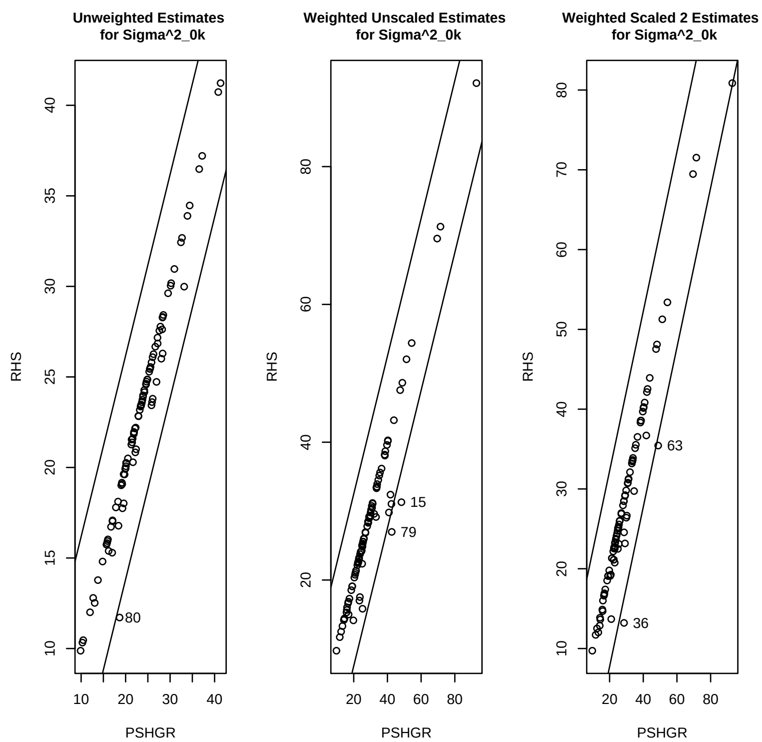


Figure 14: Comparison of PSHGR vs. RHS for Estimates from Equation 8

between 0 and 1 (or 0 and 2) while the RHS estimates are 0.25. I believe that this is a problem with the RHS estimation, however his pattern should be looked into further.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 3 contains the summary of the estimating model from Equation 7. When the estimated model matches the generating model, all of the estimation methods (PSHGR, RHS for all of unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2) have minimal bias for the β parameters. The weighted estimates have larger spreads than the unweighted estimates. In addition, the weighted unscaled estimates appear to have a larger variance than the other weighted methods for the estimation of β_2. The simulation results show minimal bias for the unweighted and weighted scaled 2 estimates, a slight positive bias for the weighted unscaled estimate and a negative bias for

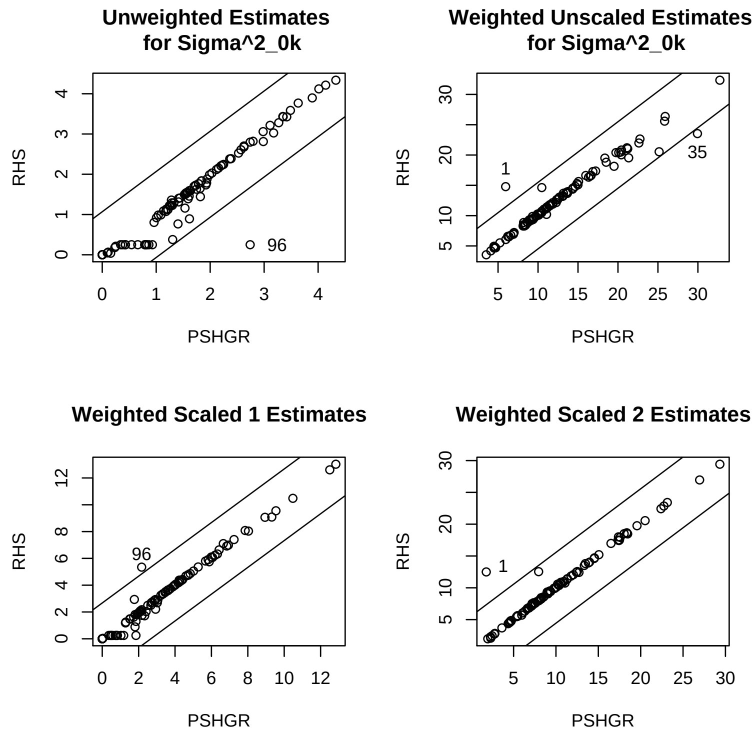


Figure 15: Comparison of PSHGR vs. RHS for Estimates from Equation 9

the weighted scaled 1 estimates. It appears that the σ_ϵ^2 parameter follows the following trend; the weighted unscaled estimates having larger negative bias, the weighted scaled 1 estimates have minimal bias and the weighted scaled 2 estimates are in between them. The unweighted estimates are also unbiased.

The middle panel of Figure 3 contains the summary of the estimating model from Equation 8. The estimated model no longer contains x_{1k}. This is a case of informative sampling as the clusters are sampled according to the size of x_{1k}. The mean of the missing $-2x_{1k}$ term would be -6 if there were not informative sampling, which would change the estimate of the intercept to be approximately 1-6=-5. However, because larger x_{1k} are oversampled, the expected value of $-2x_{1k}$ is more negative in the sample than in the population. This is reflected in the unweighted estimates with an average intercept of approximately -8. The weighted estimates help to compensate for the informative sampling, as they all have an intercept estimate of approximately -5. The estimates of β_2 are unaffected by the model misspecification and informative sampling. The variance of the missing $-2x_{1k}$ term would be 36 if there were no informative sampling. With no informative sampling, we would expect the estimate of σ^2_{0k} to be approximately 0.2+36=36.2. However, because the larger x_{1k} are oversampled, the variance of x_{1k} in the sample is less than the variance of x_{1k} in the population. This is reflected in the estimation of σ^2_{0k} because the unweighted estimates are smaller than the weighted estimates. Note that the mean of the weighted estimates is approximately 29, which is still smaller than the mean of the weighted estimates from the estimated model in Equation 5 without informative sampling, which was approximately 33. The estimates of σ^2_ϵ are not affected by this model misspecification.

The bottom panel of Figure 3 contains the summary of the estimating model from Equation 9. The estimated model no longer contains x_{2ik}. This is a case of informative sampling as the units are sampled according to the size of x_{2ik}. The mean of the missing $2x_{2ik}$ term would be 2 if there were not informative sampling, that would change the estimate of the intercept to be approximately 1+2=3. However, because larger x_{2ik} are oversampled, the expected value of $2x_{2ik}$ is larger in the sample than in the population. This is reflected in the unweighted estimates with an average intercept of approximately 9. The addition of the weights helps to compensate for the informative sampling, with intercept estimates of between 3.5 and 4.0. Note that these are still larger than the estimates from the estimation of the intercept from Equation 5 where there was no informative sampling. The estimation of β_1 is unaffected by the informative sampling and model misspecification. The variance of the missing $2x_{2ik}$ term would be 100 if there were no informative sampling. This variance is added to the estimate of σ^2_ϵ for an estimate of about 100+0.5 = 100.5 when there is no informative sampling. However, because the larger x_{2ik} are oversampled, the variance of x_{2ik} in the sample is less than the variance of x_{2ik} in the population. This smaller variance is reflected in the unweighted estimates (especially when compared to the

the results from the estimated model in Equation 5 where the unweighted estimates are larger than the weighted unscaled estimates). The estimates of σ^2_{0k} are larger than when all covariates are in the model and this is due to the smaller intra-class correlation, similar to the situation from the estimated model in Equation 5. However, the estimates of σ^2_{0k} in this simulation set are larger than the estimates from the estimated model in Equation 5.

8.1.3 Results Description for Misspecification of Random Variables – Simulation Set 5

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PHSGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. There were no differences larger than the threshold for the estimated model in Equation 11. Figure 8.1.3 contains the simulation runs in which the estimates of PSHGR and RHS are larger than the threshold for the estimated model in Equation 12. These occurred in simulation run 62 for the weighted unscaled estimate of β_0. For the estimates of σ^2_{0k}, the differences were large in the simulation run 76 for the unweighted estimates, simulation run 62 for the weighted unscaled estimates and simulation run 54 for the weighted scaled 2 estimates.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 4 contains the summary of the estimating model from Equation 11. When the estimated model matches the generating model, all of the estimation methods (PSHGR, RHS for all of unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2) have minimal bias for the β coefficients. It appears that the spread of the estimates using weighted unscaled weights is larger for the estimates of β_2 than the estimates using the other weighted methods. The unweighted estimates had a smaller spread than the weighted estimates. There is a small difference between the different weighting schemes in the estimation of the σ^2_{1k} parameter, but the differences are small compared to the differences in the σ^2_ϵ estimates. It appears that the σ^2_ϵ parameter follows the following trend; the weighted unscaled estimates having larger negative bias, the weighted scaled 1

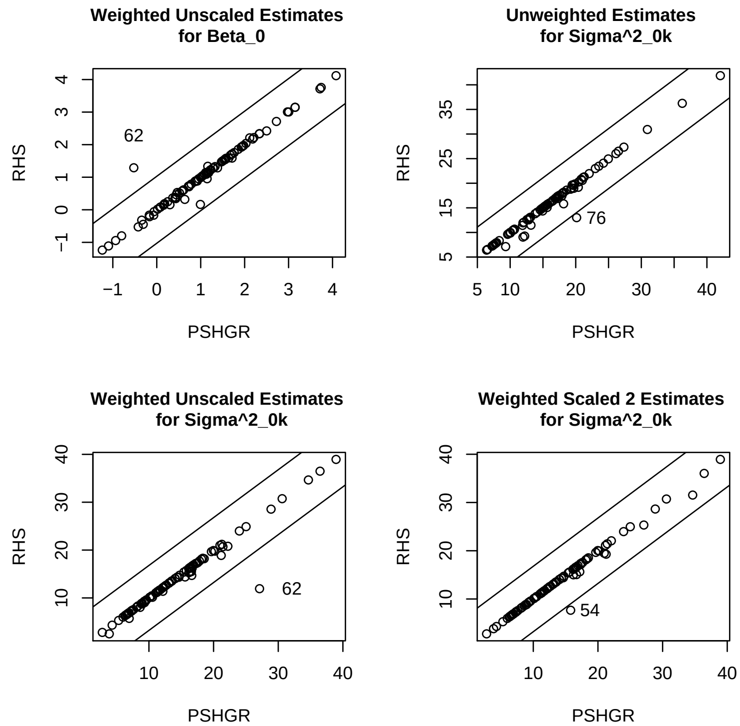


Figure 16: Comparison of PSHGR vs. RHS for Estimates from Equation 12

estimates having smaller positive bias and the weighted scaled 2 estimates being in between them.

The second row of Figure 4 misspecifies the model by removing the random slope on x_{1k}, the cluster variable, and adds a random intercept. As expected, the estimation of β_0, β_1 and β_2 are not affected by the misspecification. The random intercept includes the variation in the $U_{1k} \times x_{1k}$ variable. Recall that x_{1k} was generated as a normal random variable with mean 3 and variance 9 and U_{1k} was generated independently of x_{1k} as a normal random variable with mean 0 and variance 1. A quick simulation of 1000 sets of two simulated normal random variables set up similar to U_{1k} and x_{1k} provides variance of 18. The estimates in the figure are slightly lower (between 13.5 and 16) which follows the trend of the intercept variance having a negative bias when the ICC is large.

8.1.4 Results Description of Misspecification of Random Variables – Simulation Set 6

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. For the estimating model in Equation 14, simulation number 46 produced differences larger than the threshold in the PSHGR and RHS methods in 9 different estimates spanning all parameters and all weighting schemes. The plots to show these differences for each parameter and each scaling are not shown to conserve space. In addition, the weighted unscaled estimates of σ^2_{1k} also varied more than one threshold for simulation runs 46 and 56, see Figure 17. Of these differences, it was only the difference in the weighted unscaled estimate of σ^2_{1k} that was large enough to produce a difference in Figure 5. For the estimating model in Equation 15, there are four simulation runs whose PSHGR and RHS estimates differ by more than one threshold, as shown in Figure 18. These points correspond to simulation runs 55 and 94 for the weighted unscaled estimates of β_0 and simulation runs 39 and 94 for the weighted unscaled estimates of σ^2_{0k}.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 5 contains the summary of estimating model in Equation 14. As expected, when there is informative sampling of clusters based on the size of the random effect U_{1k}, the estimate of x_{1k} increases and the estimate of σ^2_{1k} decreases in the unweighted case. All of the weighted cases help to compensate for this informative sampling and the estimates are similar to those in Figure 4. It appears that the σ^2_ϵ parameter follows the

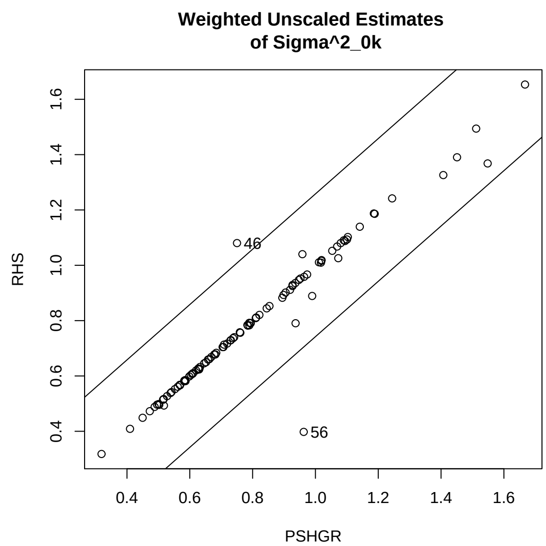


Figure 17: Comparison of PSHGR vs. RHS for Estimates from Equation 14

trend; the weighted unscaled estimates having larger negative bias, the weighted scaled 1 estimates having smaller positive bias and the weighted scaled 2 estimates being in between them.

The second row of Figure 5 misspecifies the model by removing the random slope on x_{1k}, and adding a random intercept. As expected, the estimation of β_0, β_1, β_2 and σ^2_ϵ are not affected by the misspecification and have estimates similar to the top row, though the spread of β_0 and β_1 appear to be larger. The random intercept includes the variation in the $U_{1k} \times x_k$ variable. Recall that x_{1k} was generated as a normal random variable with mean 3 and variance 9 and U_{1k} was generated independently of x_{1k} as a normal random variable with mean 0 and variance 1. A quick simulation of 1000 sets of two simulated normal random variables set up similar to U_{1k} and x_{1k} provides variance around 19. The estimates in the figure are slightly lower which follows the trend of the intercept variance having a negative bias. As can be seen by comparing this Figure to Figure 4, the estimate of the unweighted σ^2_{01} is lower than the weighted estimates, which reflects the smaller variance in the sampled U_{1k} due to the informative sampling.

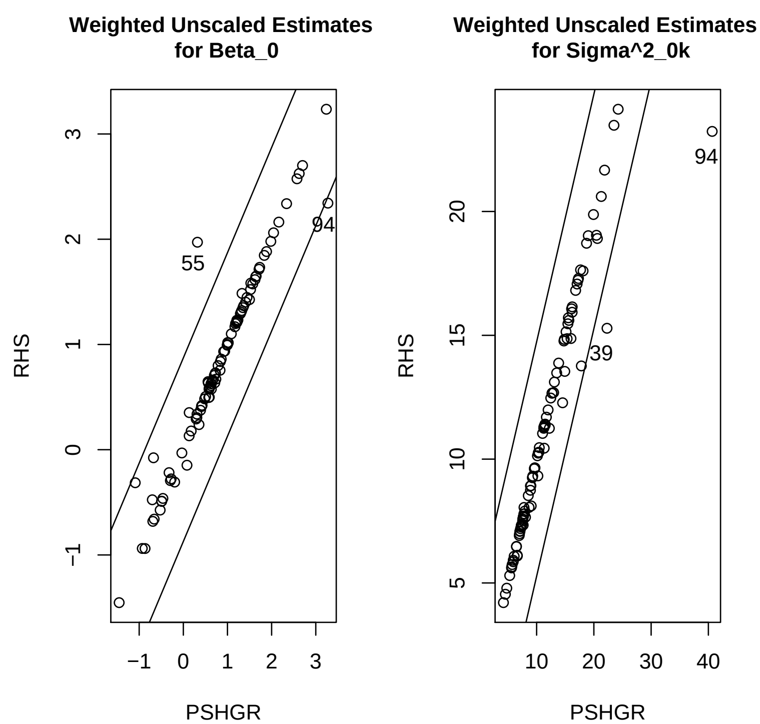


Figure 18: Comparison of PSHGR vs. RHS for Estimates from Equation 15

8.1.5 Results Description of Misspecification of Random Variables – Simulation Set 7

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. For the estimating model in Equation 17, simulation set 23 produced differences larger than the threshold in the PSHGR and RHS methods in 12 different estimates spanning all parameters and all weighting schemes. The plots to show these differences for each parameter and each scaling are not shown to conserve space. For the estimating model in Equation 18, the scaled 1 estimates of σ_ϵ^2 produced differences between PSHGR and RHS greater than the threshold in simulation runs 46 and 56, as seen in Figure 19.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 6 contains the summary of estimating model from Equation 17. When the estimated model matches the generating model, all of the estimation methods (PSHGR, RHS for all of unweighted, weighted unscaled, weighted scaled 1 and weighted scaled 2) have minimal bias and comparable quantiles. The exception to this is that the weighted unscaled estimates appear to have larger spread for the β_0 and β_1 parameters. The estimates of the σ_{2k}^2 parameter appear to be quite similar, with the exception of the unweighted estimates, that have slightly less bias. The σ_ϵ^2 parameter follows the trends; the weighted unscaled estimates having larger negative bias, the weighted scaled 1 estimates having smaller positive bias and the weighted scaled 2 estimates being in between them.

The second row of Figure 6 misspecifies the model by removing the random slope on

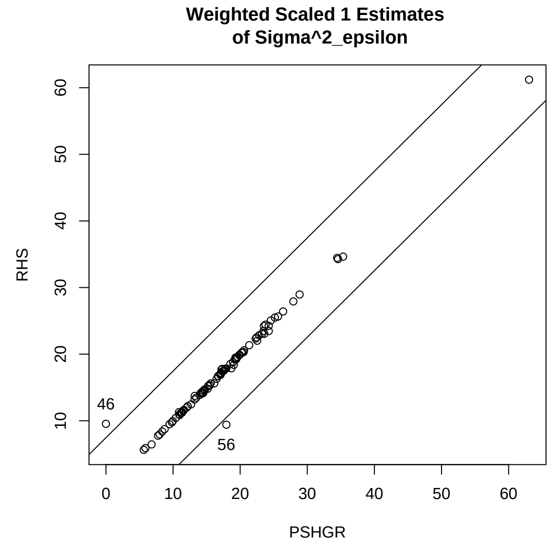


Figure 19: Comparison of PSHGR vs. RHS for Estimates from Equation 18

x_{2ik}, the unit variable, and adds a random intercept. As expected, the estimation of β_0, β_1 and β_2 are not affected by the misspecification. The random intercept includes the variation in the $U_{2k}x_{2ik}$ variable. Recall that x_{2ik} was generated as a normal random variable with mean 1 and variance 25 and U_{2k} was generated independently of x_{2ik} as a normal random variable with mean 0 and variance 0.8. We would expect a portion of the variance to go into the estimate of σ^2_ϵ and a portion to go into the σ^2_{0k}. If you condition first on the values of U_{2k}, the random error variance for that cluster will increase by $U^2_{2k} * \text{Var}(x_{2ik})$, approximately $0.8^2 * 25 = 16$. Alternatively, if we condition on x_{2ik} then the random intercept variance will increase by roughly $\bar{x}^2_{2ik}\text{Var}(U_{2k})$, approximately $1^2 * 0.8 = 0.8$. That would provide a random intercept variance of approximately 0.8+0.8=1.6, and a random error variance of approximately 0.5+16=16.5. The simulation results are consistent with these results.

8.1.6 Results Description of Misspecification of Random Variables – Simulation Set 8

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. In this simulation set, there were no differences greater than the threshold.

We next determine what we would expect the results to be for each of the estimating models. The top row of Figure 7 contains the summary of estimating model from Equation 20. As expected, when there is informative sampling of units based on the size of the random effect U_{2k}, the estimate of x_{2ik} increases and the estimate of σ^2_{2k} decreases in the unweighted case. All of the weighted cases help to compensate for this informative sampling and the estimates are similar to those in Figure 6. The weighted σ^2_{2k} estimates all have similar point estimates and ranges. The σ^2_ϵ parameter follows the trend; the weighted unscaled estimates having larger negative bias, the weighted scaled 1 estimates having smaller non-negative bias and the weighted scaled 2 estimates being in between them.

The second row of Figure 6 misspecifies the model by removing the random slope on x_{2ik}, the unit variable, and adds a random intercept. As expected, the estimation of β_0, β_1 and β_2 are not affected by the misspecification. The random intercept includes the variation in the $U_{2k}x_{2ik}$ variable. Recall that x_{2ik} was generated as a normal random variable with mean 1 and variance 25 and U_{2k} was generated independently of x_{2ik} as a normal random variable with mean 0 and variance 0.8. We would expect a portion of the variance to go into the estimate of σ^2_ϵ and a portion to go into the $\sigma^2_0 k$. If you condition first on the values of U_{2k}, the random error variance for that cluster will increase by $U_{2k}^2 * \text{Var}(x_{2ik})$, approximately $0.8^2 * 25 = 16$. Alternatively, if we condition on x_{2ik} then the variance will be roughly $\bar{x}_{2ik}^2 \text{Var}(U_{2k})$, approximately $1^2 * 0.8 = 0.8$. That would provide a random intercept variance of approximately .8+16=16.8, and a random error variance of approximately 0.5+0.8=1.3. The simulation supports these conclusions.

8.1.7 Results Description of Misspecification of Stratification Layers – Simulation Set 9

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. For the estimating model in Equation 23, there were a number of simulation runs that produced estimates the unweighted estimates of σ^2_{02k} where the differences between PSHGR and RHS greater than the threshold, as shown in Figure 20. These include simulation run 4 for the unweighted estimates, simulation runs 16 and 81 for the weighted unscaled estimates and simulation runs 19, 92 and 94 for the weighted scaled 1 estimates. The differences in the unweighted and weighted unscaled estimates are too small to be seen in Figure 8. However, the difference in the weighted scaled 1 estimates is seen due to the extreme values of the RHS estimates. In addition, the PSHGR and RHS estimates of the covariance term $\sigma^2_{01k.02k}$ were quite different, as seen in Figure 21. Further investigation is needed to better understand why the spread of the estimates are so different. The PSHGR covariance estimates are all very close to zero (less than 10^{-16} in absolute value), whereas the RHS estimates vary between approximately 3 and -3. However, the RHS weighted unscaled estimates have a few large outliers. These differences between the RHS and PSHGR estimates are clear in Figure 8. Note that the weighted scaled 1 estimates of $\sigma^2_{01k.02k}$ also follow a different pattern than the other estimates because of the extreme values of the RHS estimates. For the estimating model in Equation 24, the PSHGR and RHS weighted unscaled estimates of σ^2_{0k} for simulation run 16 are larger than the threshold, as are the estimates from simulation

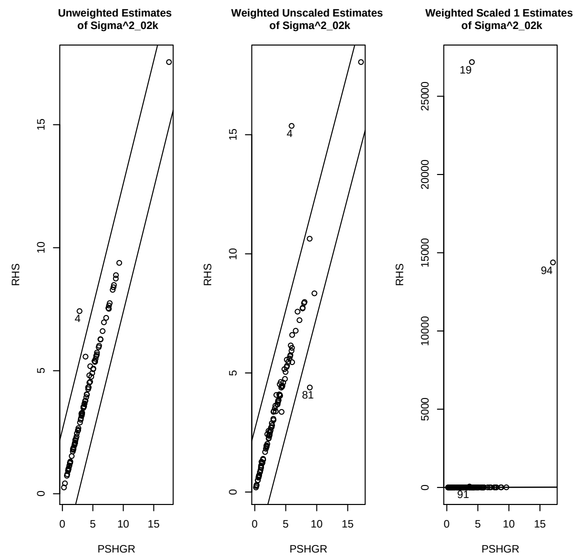


Figure 20: Comparison of PSHGR vs. RHS for Estimates from Equation 23

run 64 for the weighted unscaled estimates, as seen in Figure 22. These differences are not large enough to be seen in Figure 8. For the estimating model in Equation 26, the PSHGR and RHS weighted unscaled estimates of σ^2_{0k} for simulation runs 16, 73 and 77 are larger than the threshold, as are the estimates from simulation run 27 for the weighted scaled 2 estimates, as seen in Figure 23. These differences are not large enough to be seen in Figure 8.

We next determine what we would expect the results to be for each of the estimating models. In Figure 8, the first row shows the summary from the estimating model in Equation 23. There are two fixed effects in this regression and all estimation methods perform well. Besides the differences in the estimates between PSHGR and RHS described above, there is nothing else notable regarding the variance components. Finally, when the generating model equals the estimating model, the estimates of σ^2_ϵ follow the same trends as the previous simulations.

The second row of Figure 8 shows a summary of the results from the estimating model in Equation 24. This model is misspecified because the stratified/clustered design is estimated

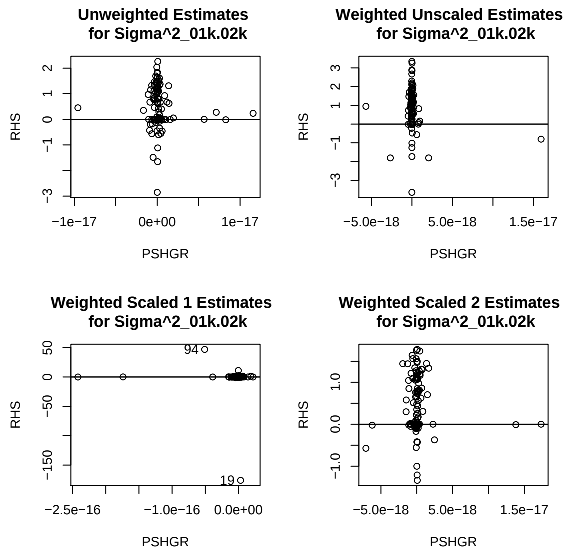


Figure 21: Comparison of PSHGR vs. RHS for Estimates of $\sigma^2_{01k.02k}$ from Equation 23

as a clustered design. Recall is no informative sampling. In this model, the two strata are being estimated as one. Since the number of elements in each strata are roughly equal, I would expect that the estimated intercept would be the average of the intercept of the two strata, in this case $(-3+5)/2 = 1$, and the graph supports this. The estimate of σ^2_ϵ is about the true value of 0.5 as the variance within each cluster should remain unchanged. The random intercept should pick up the variance associated with dropping the two strata. Note that roughly 50 sampled elements in stratum 1 have an intercept of 5, and the roughly 50 sampled elements in stratum 2 have an intercept of -3. The variance of this will be roughly $\frac{1}{100}(\sum_{i=1}^{50}(5-1)^2 + \sum_{i=1}^{50}(-3-1)^2) = 16$. The variance of 16 assumes that each strata has a fixed effect intercept. Because there are random intercepts within each stratum, the variance due to the random intercepts needs to be taken into account by increasing 16 by $\text{Var}(U_{s1} + U_{s2})/2) = 6/4 = 1.5$ to 17.5. This is consistent with the figure.

The third row of Figure 8 contains a summary from the estimated model in Equation 26. The generating model is in Equation 25. This is the same as the other two simulations in this simulation set, except that the sampling design informatively sampled clusters based

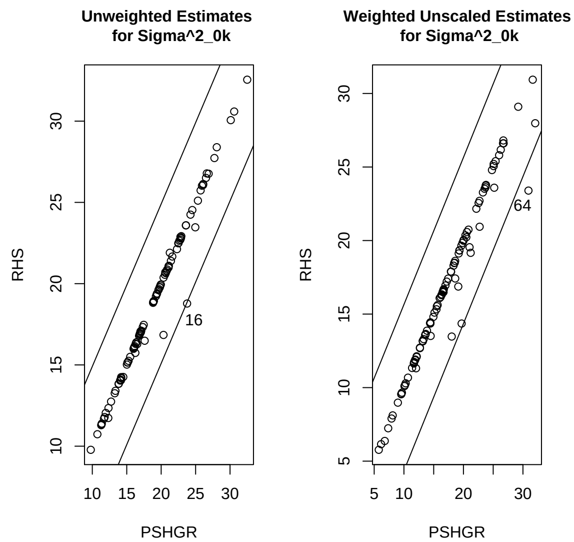


Figure 22: Comparison of PSHGR vs. RHS for Estimates from Equation 24

on the size of their random effects. When comparing the estimates of β_0, both the weighted and the unweighted estimates from Equation 26 are larger than those in Equation 24. In addition, the unweighted estimates from the estimated model in Equation 26 are larger compared to the weighted estimates than those from the estimated model in Equation 24. In addition, all of the estimates of σ^2_{0k} are smaller than the estimates from the estimating model in Equation 24. In addition the unweighted estimates of σ^2_{0k} are smaller than the weighted estimates, especially when compared to the estimates of σ^2_{0k} from the estimated model in Equation 24.

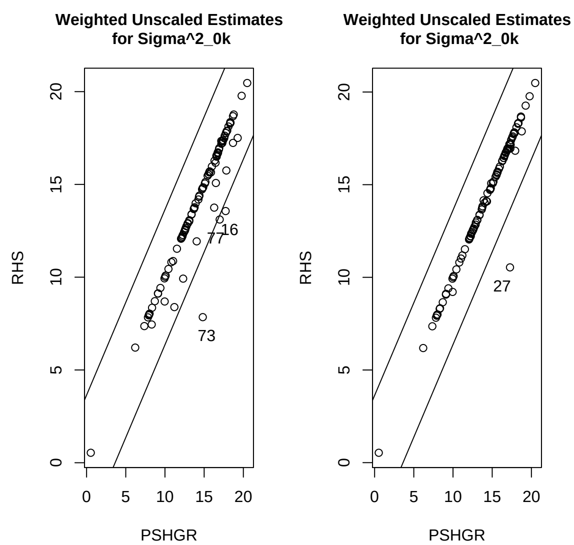


Figure 23: Comparison of PSHGR vs. RHS for Estimates from Equation 26

8.1.8 Results Description of Misspecification of Stratification Layers - Simulation Set 10

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. For the estimating model in Equation 28, there are simulation runs that produced differences between PSHGR and RHS greater than the threshold. For the estimating model in Equation 29, there are differences between PSHGR and RHS in the unweighted and weighted scaled 1 estimates of σ^2_ϵ, as shown in Figure 24. These differences are from simulation runs 5, 33, 65, 80 and 97 for the unweighted estimates and simulation runs 16, 20, 30, 35, 44, 51, 53, 57, 58, 60, 63, 64 and 95. For the weighted scaled 1 estimates of σ^2_ϵ, it is clear that most of the differences are caused when PSHGR is estimating the parameter near 0, whereas RHS is estimating the parameter between 14 and 21. I suspect this is a problem with the PSHGR computations. For the estimating model in Equation 31, there are also differences between PSHGR and RHS estimates. Figures 25 and 26 show the differences between PSHGR and RHS in the β_0, σ^2_{0k} and σ^2_ϵ parameters. Figure 25 shows that the there is a large difference between the weighted unscaled estimates of β_0 for simulation run 40, between the weighted scaled 2 estimates of σ^2_{0k} for simulation run 57, between the weighted unscaled estimates of σ^2_{0k} for runs 40 and 26, and between the weighted unscaled estimates of σ^2_ϵ for run 40. Figure 26 shows the difference between PSHGR and RHS in the weighted scaled 1 estimates of σ^2_ϵ. Similar to Figure 24, the PSHGR method has many estimates near 0, whereas the same data produced estimates between 15 and 20 for RHS. The problematic simulation

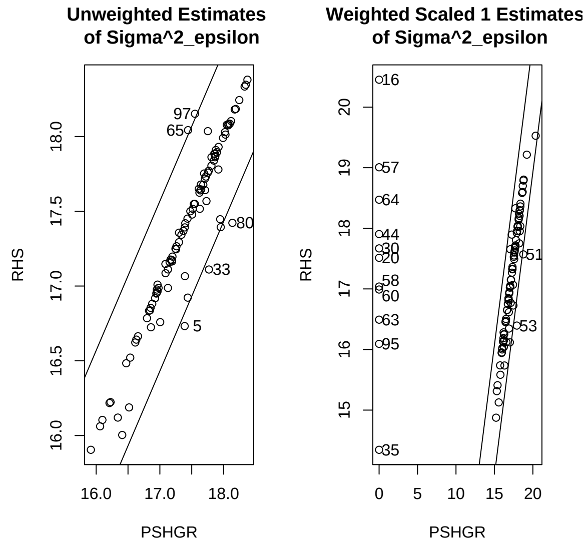


Figure 24: Comparison of PSHGR vs. RHS for Estimates from Equation 29

runs were 3, 11, 22, 37, 43, 52, 64, 69, 76, 80, 90, 92, and 97. Again, it is clear that most of the differences are caused when PSHGR is estimating the parameter near 0, whereas RHS is estimating the parameter between 14 and 21. I suspect this is a problem with the PSHGR computations.

We next determine what we would expect the results to be for each of the estimating models. In Figure 9, the first row shows the estimates of the parameters when the estimating model from Equation 28 matches the generating model. All of the weighting methods estimate the β parameter well. The unweighted estimates have a smaller spread. The spread for the weighted unscaled estimation for β_1 appears to be wider than the other weighted methods. The estimate of σ^2_{0k} appears to be the similar across different weighting methods, likely due to the higher intra-class correlation. The pattern of the estimates for σ^2_ϵ is the same as in previous simulations.

The second row of Figure 9 misspecifies the model by removing the stratification, so that the stratified/clustered design is estimated as a clustered design, as detailed in Equation 29. In this second row, the clusters are sampled proportional to an independent random

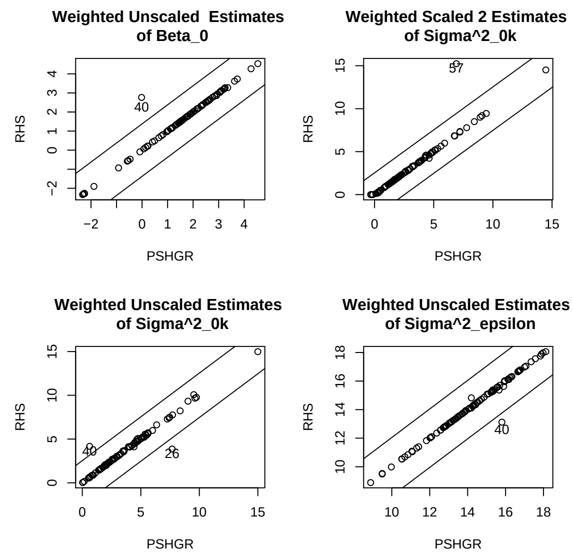


Figure 25: Comparison of PSHGR vs. RHS for Estimates from Equation 31

variable (non-informatively). Here, I would expect the variance of the random intercept to remain the same, and the random error term variance, σ_ϵ^2 to absorb the variance from not including the stratification in the model. Note that roughly half sampled elements in a cluster are in stratum 1 with an intercept of 5, and the roughly half sampled elements in a cluster are in stratum 2 with an intercept of -3. If n_k is the number of elements in cluster k, then the variance of the error term will be roughly $\frac{1}{n_k}(\sum_{i=1}^{n_k/2}(5-1)^2+\sum_{i=1}^{n_k/2}(-3-1)^2) = 16$. Adding this to the original random error of 0.5 gives an estimated value of σ_ϵ^2 of about 16.5, as seen in the figure. Because the intra-class correlation is smaller now due to the increase in the random error variance, the estimates of σ_{0k}^2 are exhibiting the behavior of the previous simulations with a low intra-class correlation.

The third row in Figure 9 sampled the clusters informatively, porportional to the size of the random effect (U_{0k}), as detailed in Equation 31. Because of this, the estimate of the random intercept is larger in the unweighted case and the estimate of the variance of the random intercept is smaller The smaller variance in the unweighted estimate can be seen by comparing the unweighted estimate of the random intercept in the second row of Figure

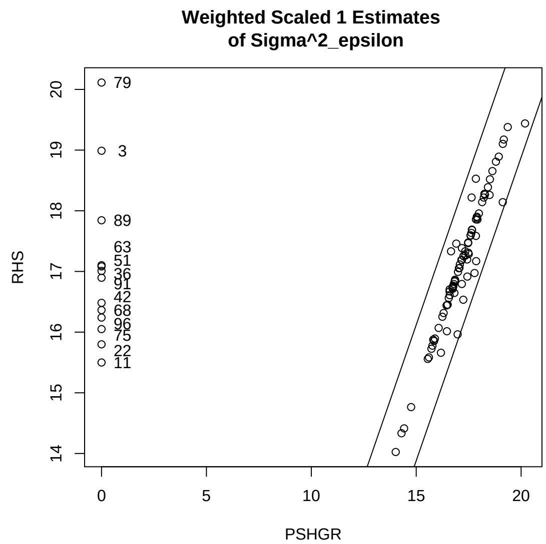


Figure 26: Comparison of PSHGR vs. RHS for Estimates from Equation 31 (cont)

9 with the unweighted estimate of the random intercept of the third row of the same figure. These are corrected with the weighted estimates. The estimate of σ_ϵ^2 remains unchanged, as expected.

8.1.9 Results Description of Misspecification of Stratification Layers - Simulation Set 11

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. Figures 27, 28 and 29 contain graphs of the estimates from the simulation runs whose difference between the PSHGR and RHS estimates is larger than the thresholds from the estimating model in Equation 33. Note that simulation run 10 did not converge for the RHS weighted unscaled estimates. From Figure 27, we see that PSHGR and RHS methods differed for simulation run 71 in the weighted scaled 1 estimates of β_0, β_1 and σ^2_{01k}. The effect of the large difference from run 71 can be seen in Figure 10 in the difference between the means of the RHS and PSHGR scaled 1 estimates of β_0 and σ^2_{01k}. In addition, the PSHGR and RHS weighted unscaled estimates of β_1 differed by more than the threshold in simulation run 60.

Figure 28 contains the graphs of PSHGR vs. RHS estimates for the σ^2_{02k} parameter. All of the weighting methods contained simulation runs that produced large differences between the PSHGR and RHS estimates. For the unweighted estimates, simulation runs 17 and 23 produced large differences. Note that in Figure 10, the mean of RHS unweighted estimates of σ^2_{02k} is larger than the spread of the 0.025 to 0.975 quantiles. This is due to the large value from simulation run 17. For the weighted unscaled estimates, simulation runs 11, 13, 26, 32, 36, 38, 51, 60, 65, 89, and 97 produced large differences. In Figure 10, these large differences are reflected as a much larger spread and mean for the RHS weighted unscaled estimates than for the PSHGR weighted unscaled estimates. For the

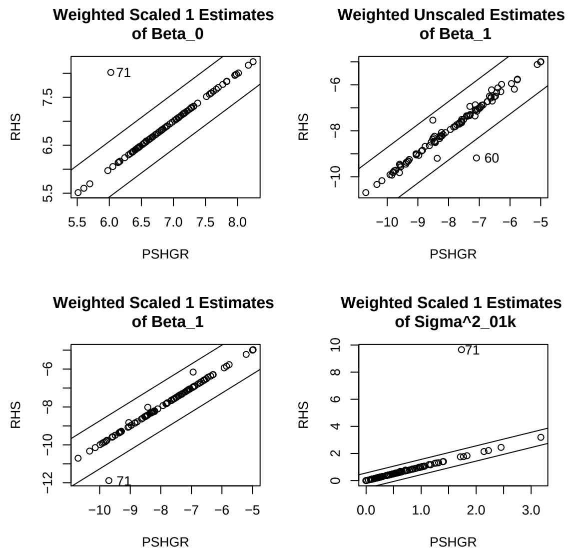


Figure 27: Comparison of PSHGR vs. RHS for Estimates from Equation 33

weighted scaled 1 estimates, simulation runs 2 and 25 produced large estimates. Finally, for the weighted scaled 2 estimates, simulation runs 13, 45 and 63 produced large differences. These differences are shown in Figure 10 in that the mean of the RHS estimate is not on the graph. The large value(over 80,000) from simulation run 13 for RHS causes the RHS mean to be larger than the scale printed in the figure.

Figure 29 contains the graphs of PSHGR vs. RHS estimates for the $\sigma^2_{01k.02k}$ parameter. This trend is similar to the estimated covariance term from Equation 23 seen in Figure 21. The PSHGR estimates are showing a small amount of variablility (note that the scales on the x-axis are no larger than $\pm 4 \times 10^{-17}$). The RHS scales are roughly ± 3, except for the weighted scaled 1 estimates where the RHS has some large estimates, around 50 and -170 and the weighted scaled 2 estimates about 250. In Figure 10 it is clear that the spread of the PSHGR estimates is smaller than the RHS estimates. The larger estimates of the RHS scaled 1 estimates is reflected in a larger spread in the figure. In addition, the large value (about 250) of the RHS weighted scaled 2 estimate is causing the mean to be large in Figure 10.

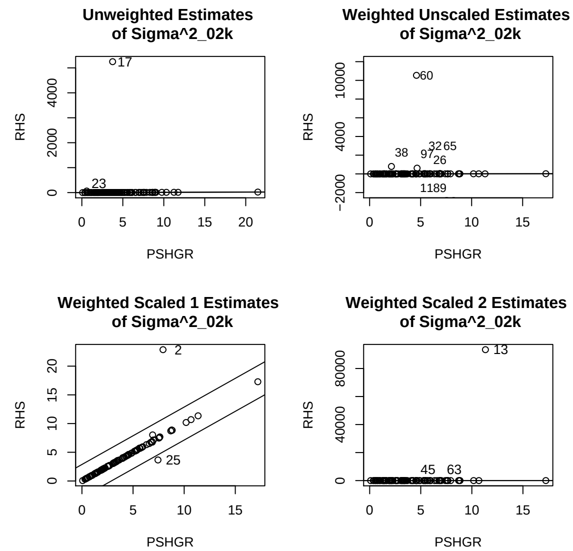


Figure 28: Comparison of PSHGR vs. RHS for Estimates of σ^2_{02k} from Equation 33

Figure 30 contains the graphs of PSHGR vs. RHS estimates for the σ^2_{0k} parameter from the estimating model in Equation 34. In general, these simulation show many differences between PSHGR and RHS in this parameter, except there are no differences for the weighted scaled 1 estimates. For the unweighted estimates, simulation runs 2, 3, 16, 21, 28, 35, 39, 53, 59, 67, 74, 75, and 87 produced estimates with differences larger than the threshold. For the weighted unscaled estimates, simulation runs 8, 27, 48, 52, 53, 62, 76, 79, 82, 84, and 93 produced estimates with differences larger than the threshold. For the weighted scaled 2 estimates, simulation runs 32, 68, 71, 78, and 89 produced estimates with differences larger than the threshold. These differences can be seen in Figure 10 in the comparison of the PSHGR and RHS unweighted, weighted unscaled and weighted scaled 2 estimates.

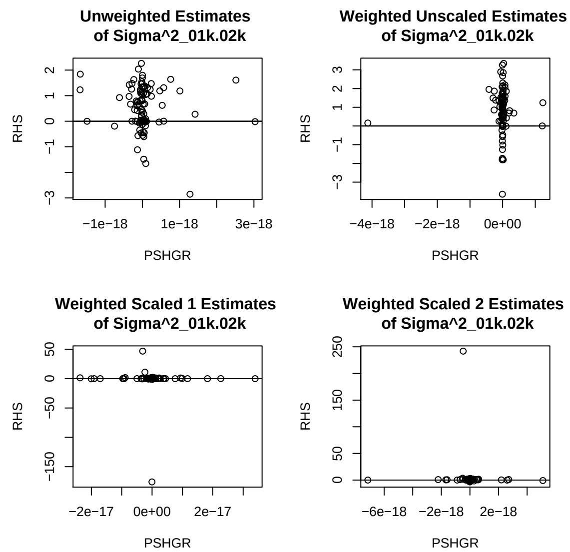


Figure 29: Comparison of PSHGR vs. RHS for Estimates of $\sigma^2_{01k.02k}$ from Equation 33

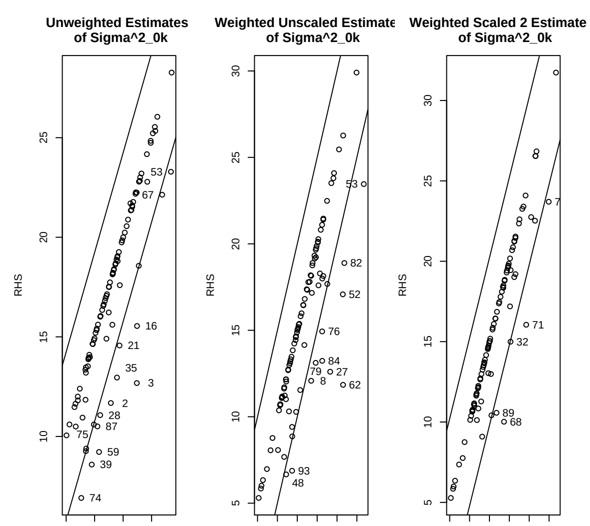


Figure 30: Comparison of PSHGR vs. RHS for Estimates from Equation 34

There are many issues with the estimation from the estimated model in Equation 35. The PSHGR method produces estimates in only 75 of the 100 simulation runs. This is mostly due to not being able to invert matrices needed for the computation of V^{-1}. This needs to be further investigated. The simulation runs that contained computation problems are 4, 6, 7, 19, 20, 23, 26, 30, 32, 34, 36, 41, 43, 45, 58, 63, 65, 68, 72, 74, 80, 81, 86, 90, and 94. In addition, there are a number of PSHGR runs that did not converge within 500 iterations for the weighted scaled 1 estimates, including runs 12, 14, 15, 21, 27, 31, 54, 55, 62, 67, 77, 83, 91, 93, 97, 98, 99, and 100. The RHS method did not converge for simulation run 6 for the scaled 1 estimates and for simulation run 71 for the scaled 2 estimates. As can be seen in Figures 31 to 36, the estimation from this model produces many differences between PSHGR and RHS.

Figure 31 contains the graphs of PSHGR vs. RHS estimates for the estimate of β_0. The weighted unscaled estimates produced differences between PSHGR and RHS larger than the threshold for simulation run 75. The weighted scaled 1 estimates produce differences between PSHGR and RHS larger than the threshold for simulation runs 28, 50, and 75. The weighted scaled 2 estimates produced differences between PSHGR and RHS larger than the threshold for simulation run 37. The differences between the weighted scaled 1 estimates of PSHGR and RHS can be seen in Figure 10 as the PSHGR 0.025 quantile and mean are lower than the corresponding RHS values. The other differences are too small to notice on the figure.

Figure 32 contains the graphs of PSHGR vs. RHS estimates for the estimate of β_1. The weighted unscaled estimates produced differences between PSHGR and RHS larger than the threshold for simulation run 75. The weighted scaled 1 estimates produce differences between PSHGR and RHS larger than the threshold for simulation runs 28 and 75. The weighted scaled 2 estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 37 and 62. The differences are reflected in Figure 10 by PSHGR having a larger quantile than RHS for the scaled 1 estimate of β_1 and PSHGR

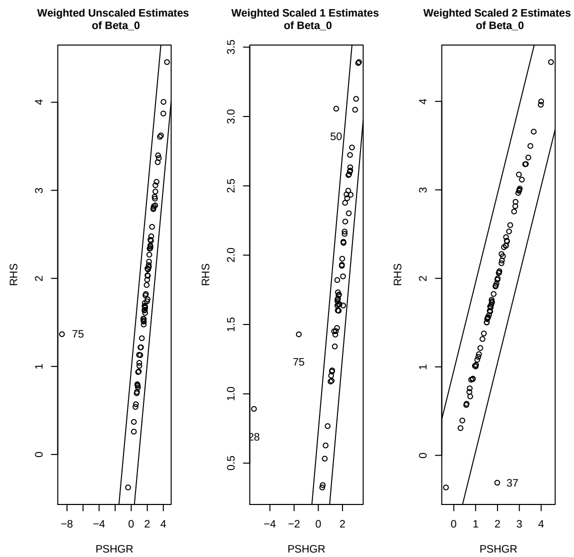


Figure 31: Comparison of PSHGR vs. RHS for Estimates of β_0 from Equation 35

having a smaller 0.025 quantile than RHS for the weighted unscaled estimates of β_1. The other differences are too small to notice on the figure.

Figure 33 contains the graphs of PSHGR vs. RHS estimates for the estimate of σ^2_{01k}. The unweighted estimates produced differences between PSHGR and RHS larger than the threshold for simulation run 2. The weighted unscaled estimates produce differences between PSHGR and RHS larger than the threshold for simulation runs 2, 3, 10, 15, 16, 21, 22, 25, 39, 40, 50, 53, 79, 84, 85, and 97. The weighted scaled 2 estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 3, 10, 13 and 47. The other differences are too small to notice on the figure.

Figure 34 contains the graphs of PSHGR vs. RHS estimates for the estimate of σ^2_{02k}. The unweighted estimates produced differences between PSHGR and RHS larger than the threshold for simulation run 71. The weighted unscaled estimates produce differences between PSHGR and RHS larger than the threshold for simulation runs 9, 28 and 57. The weighted scaled 2 estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 37 and 42. These differences are reflected in Figure 10

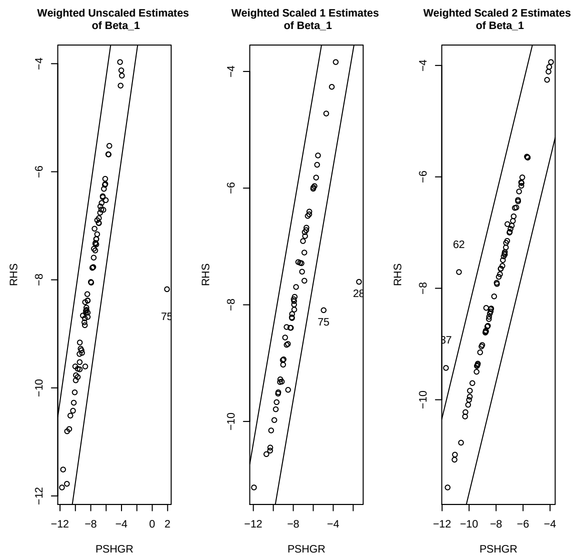


Figure 32: Comparison of PSHGR vs. RHS for Estimates of β_1 from Equation 35

because the RHS weighted scaled 2 mean is so large (due to the simulation run 37 having an estimate of 2500) that it is not printed on the plot for σ^2_{01k}. These differences are reflected in Figure 10 by RHS having a larger 0.975 quantile for the weighted unscaled estimates than PSHGR. Also, the RHS simulation runs 3, 10, 13 and 47 cause the RHS 0.975 quantile to be larger than the PSHGR corresponding quantile for the weighted scaled 2 estimates. The mean of the RHS weighted scaled 2 estimate of σ^2_{02k} is printed off of the scale of the graph on Figure 10. The other differences are too small to notice on the figure.

Figure 35 contains the graphs of PSHGR vs. RHS estimates for the estimate of $\sigma^2_{01k.02k}$. In this figure, we see the same trends as we did in Figures 21 and 29. The variation in the PSHGR estimates is very small, with the largest variation being approximately 2^{-13}. The RHS estimates have more spread, with the weighted scaled 2 estimates containing two large estimates around 3000 and 6000. This pattern should be looked into further. These differences are reflected in Figure 10 by the small ranges for the PSHGR estimates and the larger ranges for the RHS weighted unscaled and weighted scaled 2 estimates. In addition, the mean of the RHS weighted scaled 2 estimates is so large (due to the estimates of 6000

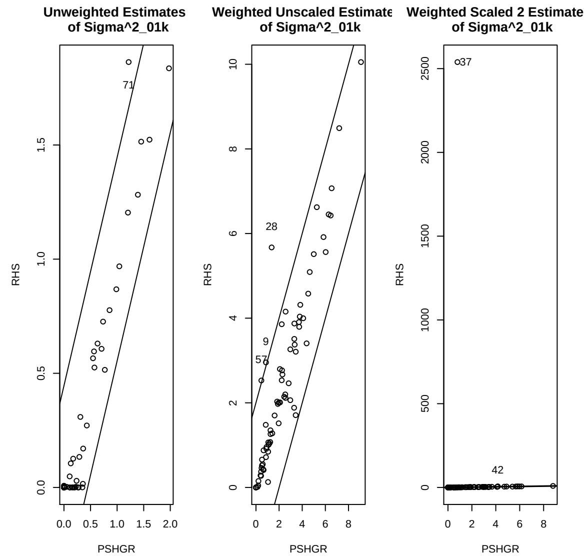


Figure 33: Comparison of PSHGR vs. RHS for Estimates of σ^2_{01k} from Equation 35

and 3000) that it is not printed on the range of the graph. The other differences are too small to notice on the figure.

Figure 36 contains the graphs of PSHGR vs. RHS estimates for the estimate of σ^2_{ϵ}. The unweighted estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 2, 16, 35, 37, 59 and 71. The weighted unscaled estimates produce differences between PSHGR and RHS larger than the threshold for simulation run 75. The weighted scaled 1 estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 28, 37, 40, 50, 73, 75 and 76. The differences are reflected in Figure 10 by small value of the 0.025 PSHGR quantile of the weighted scaled 1 estimates. The other differences are too small to notice on the figure.

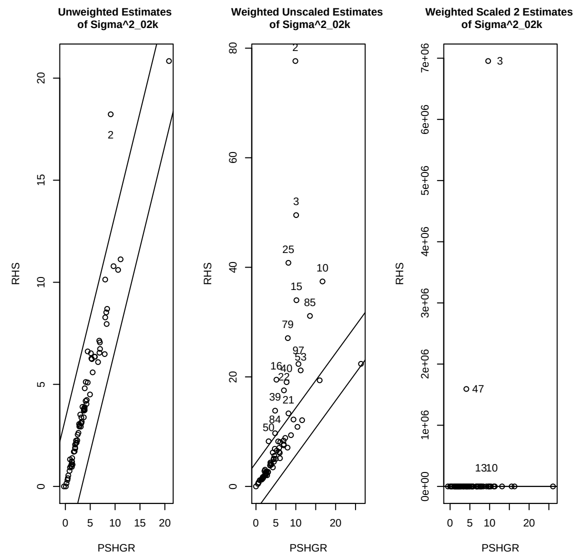


Figure 34: Comparison of PSHGR vs. RHS for Estimates of σ^2_{02k} from Equation 35

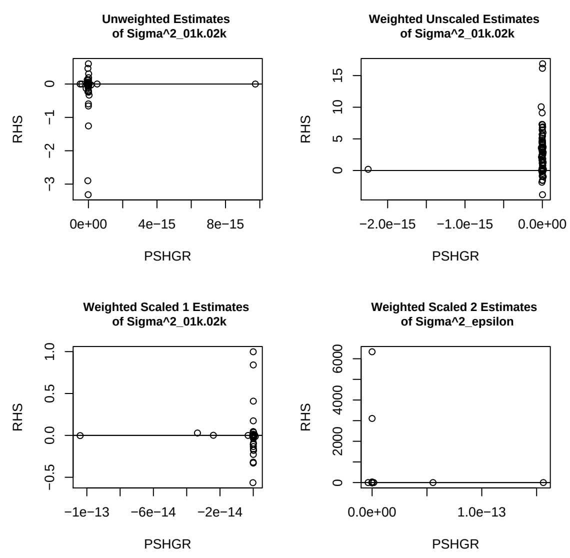


Figure 35: Comparison of PSHGR vs. RHS for Estimates of $\sigma^2_{01k.02k}$ from Equation 35

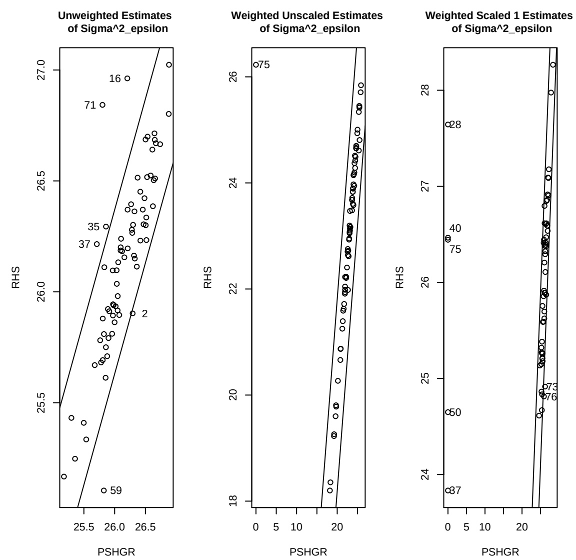


Figure 36: Comparison of PSHGR vs. RHS for Estimates of σ_ϵ^2 from Equation 35

Figure 37 contains the estimates of σ^2_{0k} from the estimated model in Equation 36. The weighted unscaled estimates produced differences between PSHGR and RHS larger than the threshold for simulation runs 30 and 65. These differences are too small to notice on the Figure 10.

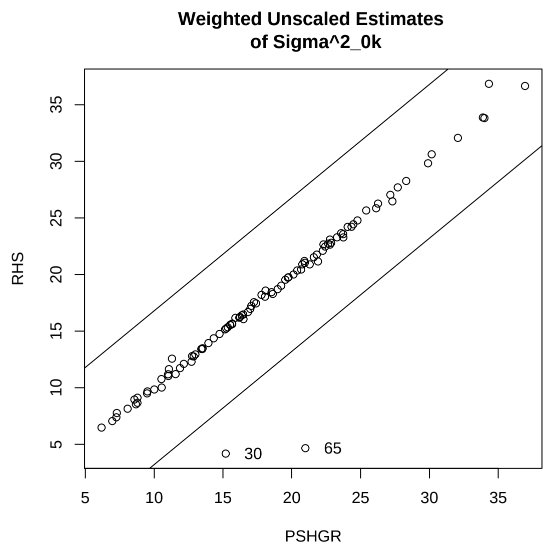


Figure 37: Comparison of PSHGR vs. RHS for Estimates of σ^2_{0k} from Equation 36

We next determine what we would expect the results to be for each of the estimating models. In Figure 10, the first row shows the estimates of the parameters when the estimating model matches the generating model. Generally, the estimation does well with the exception of the large estimates of RHS outlined above.

The second row of Figure 10 misspecifies the model by removing the top level of stratification, so that the stratified/clustered/stratified design is estimated as a clustered/stratified design. In this second row, the clusters are sampled proportional to an independent random variable (non-informatively). In this model, the two top level strata are being estimated as one. Since the number of elements in each strata are roughly equal, I would expect that the estimated intercept would be the average of the intercept of the two strata, in this case $(-3 + 5)/2 = 1$. However, now the reference point for the intercept is the lower level stratum 1, that has an intercept of two. Thus, the intercept is now 1+2=3, and the graph supports this. The second level stratum level two coefficient has not changed. The estimate of σ^2_ϵ is about the true value of 0.5 as the variance within each cluster should remain unchanged. The random intercept should pick up the variance associated with dropping the two strata. Note that roughly 50 sampled elements in stratum 1 have an intercept of 5, and the roughly 50 sampled elements in stratum 2 have an intercept of -3. The variance of this will be roughly $\frac{1}{100}(\sum_{i=1}^{50}(5-1)^2 + \sum_{i=1}^{50}(-3-1)^2) = 16$. In addition, there is the variance from the random intercepts. Here, the random intercepts are $\text{var}((U_{s1} + U_{s2})/2) = 6/4 = 1.5$. This would lead to the overall random intercept with a variance of 16+1.5=17.5. This is consistent with the figure.

The third row of Figure 10 misspecifies the model by removing the second level of stratification, so that the stratified/clustered/stratified design is estimated as a stratified/clustered design. In this second row, the clusters are sampled proportional to an independent random variable (non-informatively). The intercept now represents the top level of stratification (averaged over the bottom level of stratification). The average of the bottom level of stratification is $(2-8)/2 = -3$. Thus, the intercept should be $5-3 = 2$, as shown in the graph. Note that the variance components for the RHS weighted scaled 2 estimation method have large spreads. This is due to two simulations creating large outliers for these estimates. I would expect the variance of the random intercept to remain the same, and the random error term variance , σ^2_ϵ to absorb the variance from not including the stratification in the model. Note that roughly half sampled elements in a cluster are in the first lower level stratum with an intercept of 2, and the roughly half sampled elements in a cluster are in the second lower level stratum with an intercept of -8. If n_{ks} is the number of elements in cluster k where S2=1 (or S2=2, as the strata are roughly equally sized), then the variance of the error term will be roughly $\frac{1}{2*n_{ks}}(\sum_{i=1}^{n_k s}(2+3)^2 + \sum_{i=1}^{n_k}(-8+3)^2) = 25$. Adding this to the original random error of 0.5 gives an estimated value of σ^2_ϵ of about 25.5, as seen in the figure.

The fourth row of Figure 10 misspecifies the model by removing the all levels of stratification, so that the stratified/clustered/stratified design is estimated as a clustered design. In this second row, the clusters are sampled proportional to an independent random variable (non-informatively). The intercept now represents the average across all strata. We know that s1=1 has an intercept of 5, s1=2 has an intercept of -3, S2=1 has an intercept of 2 and S2=2 has an intercept of -8. Averaging these (as they all have roughly the same number of people) provides a grand intercept of -1, as indicated by the figure. Removing the lower level of stratification (the S2 level) will increase the estimate of σ_ϵ^2. The increase will be by 25, as indicated in the description in the above paragraph. Thus, the estimated σ_ϵ^2 should be 25+0.5 = 25.5, which is supported by the figure. In addition, the variance induced by removing the top level of stratification is put into the random intercept. As described in the previous two paragraphs, the variance of the random intercept should be about 16.5, as represented in the figure.

8.1.10 Results Description of Misspecification of Clustering Layers - Simulation Set 12

We want to flag if there are large differences between the PSHGR and RHS estimates for a given iteration. To do this, the standard deviation of the parameter estimate over the 100 iterations is obtained separately for the PSHGR and the RHS estimates. The smaller of these standard deviations is used as a threshold to flag "large" differences between PSHGR and RHS estimates. For each iteration, the difference between the PSHGR and the RHS estimates is compared to the threshold to identify estimates where the difference is greater than one standard deviation. Unless otherwise mentioned, the difference between the PSHGR and RHS estimates is less than the threshold. Figure 38 contains the estimates where PSHGR and RHS are larger than the threshold from the estimating model in Equation 38. For the weighted scaled 2 estimates of $\sigma^2_{0k_1}$, PSHGR and RHS estimates have large differences for the simulation runs 13, 18 and 91. For the weighted unscaled estimates of σ^2_ϵ, the simulation run 97 produced large differences between PSHGR and RHS. For the weighted scaled 1 estimates of σ^2_ϵ, the simulation runs 26, 27, 42, 53, 54, 55, 81, 93, and 97 produced large differences between PSHGR and RHS. For the weighted scaled 2 estimates of σ^2_ϵ, the simulation runs 2, 5, 8, 10, 13, 18, 25, 41, 42, 46, 48, 49, 50, 54, 55, 59, 61, 63, 65, 68, 69, 73, 76, 78, 82, 92, and 98 produced large differences between PSHGR and RHS.

Figure **??** contains the estimates of $\sigma^2_{0k_1k_2}$ from the estimating model in Equation 39. This figure shows that simulation run 68 caused large differences between the PSHGR and RHS weighted unscaled, weighted scaled 1 and weighted scaled 2 estimates.

We next determine what we would expect the results to be for each of the estimating models. In Figure 11, the first row shows the estimates of the parameters when the bottom layer of clustering is removed. With this the variance of the $U_{0k_1k_2}$ term is put into the estimate of σ^2_ϵ, that becomes 1.5. There is some negative bias in the estimate of σ^2_{0k1}, due to the large intra-class correlation ($4/5.5 = 0.73$).

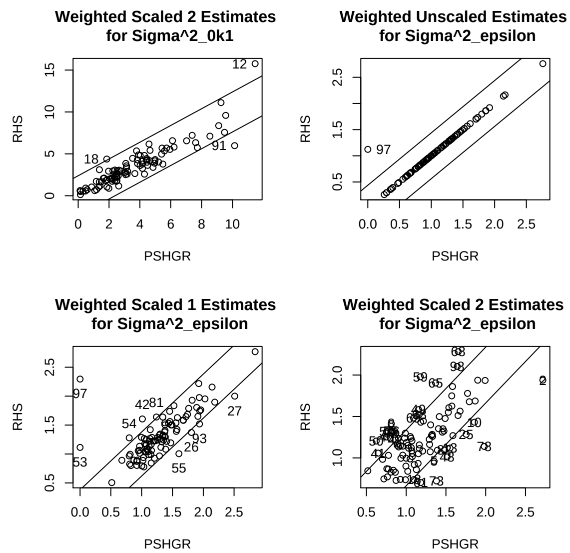


Figure 38: Comparison of PSHGR vs. RHS for Estimates from Equation 38

The second row shows the estimates of the parameters when the top layer of clustering is removed. The variance of $\sigma^2_{0k_1}$ should be put into the estimate of $\sigma^2_{0k_1k_2}$ to produce an estimate of 6. There is negative bias again, likely due to the large intra-class correlation (6/6.5=0.923).

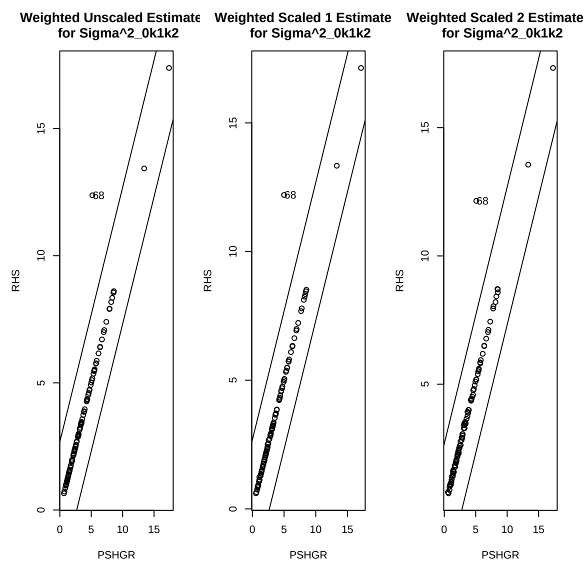


Figure 39: Comparison of PSHGR vs. RHS for Estimates from Equation 39

8.2 Negative Variance Components

There are situations in which the variance components are estimated to be negative. For example, in the case of a balanced random intercept model, say $y_{ik} = \beta_0 + \beta_{xk} x_k + \beta_{xij} x_{ij} + U_{0k} + \epsilon_{ik}$, the closed form estimate of the MLE of σ^2_{0k} in the unweighted case is $\frac{SSA}{k_S n_1} - \frac{\hat{\sigma}^2_\epsilon}{n_1}$, where k_S is the number of sampled clusters and n_1 is the number of elements sampled per cluster. In this case $SSA = \sum_k n_1[(\bar{y}_{\cdot k} - \bar{x}_{\cdot k}\beta) - (\bar{y}_{\cdot\cdot} - \bar{x}_{\cdot\cdot}\beta)]^2$, where the $\cdot$ in the subscript defines the variable being averaged over. There are cases when $\frac{\hat{\sigma}^2_\epsilon}{n_1}$ will be less than the term with SSA, resulting in a negative estimate for σ^2_{0k}. As is described in Searle et al. (1992) §3.7, when this occurs, the MLE of σ^2_{0k} becomes zero, and the estimate for σ^2_ϵ is adjusted. In this case, the $E(\hat{\sigma}^2_{0k}) = (1-p)E(\hat{\sigma}^2_{0k} | \hat{\sigma}^2_{0k} \geq 0)$ where p is the probability that $\hat{\sigma}^2_{0k}$ is negative. The density for the conditional distribution is not tractible, making the expected value difficult to obtain but can be estimated empirically in the simulations in this chapter. From Searle et al. (1992), p can be computed as $p = \Pr(\mathcal{F}^{K(N_1-1)}_{K-1} > (1 - 1/K)(1 + n\frac{\sigma^2_{0k}}{\sigma^2_\epsilon}))$, where $\mathcal{F}^{K(N_1-1)}_{K-1}$ is a random variable with an F distribution with $K(N_1 - 1)$ and $K - 1$ degrees of freedom.

This situation of negative estimated variance components occurs in some simulations, and will be noted as necessary. The simulations in this chapter are not balanced, and so the adjustment to the estimate of σ^2_ϵ is not computed. However Searle et al. (1992) show that the estimate of σ^2_ϵ without the adjustment (which are computed in the simulations) form an upper bound on the MLE.

8.3 RHS Sensitivity to the Number of Quadrature Points

To further investigate the differences between RHS and PSHGR, some of the simulation runs where the methods produce different estimates are examined. Specifically, various points from Figure 24 were run for a range of iteration points for RHS in the `gllamm()` function. The first point examined is in the first panel of Figure 24, an unweighted estimate of σ^2_{0k} from simulation run 2. The PSHGR and RHS results from a number of iteration points ranging from 15 to 30 are in Table 9. The table shows that the $\hat{\beta}_0, \hat{\beta}_2$ and $\hat{\sigma}^2_\epsilon$ are mostly unaffected by the iteration points . The estimates of $\hat{\sigma}^2_{0k}$ are quite sensitive, ranging from 9.32 to 17.83. Note from the log likelihood values, the maximum occurs at the parameter estimates from PSHGR. There are a number of iteration points that provide RHS estimates similar to the PSHGR estimates. Note that for 20 iteration points, the method did not converge. When the simulations were run for simulation set 11, the first converged simulation starting with 15 iteration points was chosen. Note that increasing the number of iteration points does not produce a monotonic increase in the log likelihood, as the lowest log likelihood occurred with 21 iteration points.

Method (Number of Iteration Points)	$\hat{\beta}_0$	$\hat{\beta}_2$	$\hat{\sigma}_\epsilon^2$	$\hat{\sigma}_{0k}^2$	Log Likelihood
PSHGR	3.24	-10.02	0.53	17.85	-475.04
RHS (15)	3.24	-10.02	0.53	11.68	-475.56
RHS (16)	3.25	-10.02	0.53	17.81	-475.04
RHS (17)	3.24	-10.02	0.53	17.83	-475.04
RHS (18)	3.23	-10.04	0.53	15.09	-475.15
RHS (19)	3.24	-10.02	0.53	17.83	-475.04
RHS (20)	NA	NA	NA	NA	NA
RHS (21)	3.24	-10.03	0.53	9.32	-476.39
RHS (22)	3.24	-10.02	0.52	15.06	-475.12
RHS (23)	3.24	-10.02	0.52	15.05	-475.12
RHS (24)	3.24	-10.02	0.53	17.85	-475.04
RHS (25)	3.24	-10.02	0.53	17.84	-475.04
RHS (26)	3.24	-10.02	0.53	17.85	-475.04
RHS (27)	3.25	-10.02	0.53	17.85	-475.04
RHS (28)	3.24	-10.02	0.53	17.84	-475.04
RHS (29)	3.24	-10.02	0.53	17.84	-475.04
RHS (30)	3.24	-10.02	0.53	17.85	-475.04

Table 9: Differences between RHS and PSHGR Estimated Parameters for Unweighted Estimates from Simulation Run 2 from Simulation Set 11, Estimating Model from Equation 34

Method (Number of Iteration Points)	$\hat{\beta}_0$	$\hat{\beta}_2$	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_{0k}$	Log Likelihood
PSHGR	2.62	-10.02	0.51	28.40	-470.85210
RHS (15)	2.28	-10.02	0.51	23.29	-470.98088
RHS (16)	2.61	-10.02	0.51	28.39	-470.85214
RHS (17)	2.71	-10.02	0.51	25.99	-470.87701
RHS (18)	2.61	-10.02	0.51	28.47	-470.85211
RHS (19)	2.65	-10.02	0.51	28.31	-470.85234
RHS (20)	2.62	-10.02	0.51	28.39	-470.85210
RHS (21)	2.62	-10.02	0.51	28.32	-470.85214
RHS (22)	2.60	-10.04	0.52	22.30	-471.06292
RHS (23)	2.83	-10.02	0.51	26.20	-470.87748
RHS (24)	2.62	-10.02	0.51	28.40	-470.85210
RHS (25)	2.61	-10.02	0.51	28.43	-470.85213
RHS (26)	2.62	-10.02	0.51	28.40	-470.85210
RHS (27)	2.61	-10.02	0.51	28.44	-470.85212
RHS (28)	2.61	-10.02	0.51	28.44	-470.85212
RHS (29)	2.59	-10.02	0.51	28.54	-470.85231
RHS (30)	2.62	-10.02	0.51	28.36	-470.85211

Table 10: Differences between RHS, and PSHGR Estimated Parameters for Unweighted Estimates from Simulation Run 53 from Simulation Set 11, Estimating Model from Equation 34

The next point examined is in the first panel of Figure 24, an unweighted estimate of σ^2_{0k} from simulation run 53. The PSHGR and RHS results from a number of iteration points ranging from 15 to 30 are in Table 10. The table shows that the $\hat{\beta}_2$ and σ^2_ϵ are mostly unaffected by the number of iteration points. The estimates of $\hat{\beta}_0$ do vary between 2.28 and 2.83. The $\hat{\sigma}^2_{0k}$ are quite sensitive, ranging from 23.30 to 28.54. Note from the log likelihood values, the maximum occurs at the parameter estimates from PSHGR. There are a number of iteration points that provide RHS estimates similar to the PSHGR estimates. When the simulations were run for simulation set 11, the first converged simulation starting with 15 iteration points was chosen. Note that increasing the number of iteration points does not produce a monotonic increase in the log likelihood, as the lowest log likelihood occurred with 22 iteration points.

Method (Number of Iteration Points)	$\hat{\beta}_0$	$\hat{\beta}_2$	$\hat{\sigma}^2_\epsilon$	$\hat{\sigma}^2_{0k}$	Log Likelihood
PSHGR	2.20	-10.13	0.52	31.67	NA
RHS (15)	1.89	-10.16	0.48	23.46	-1182.6300
RHS (16)	1.98	-10.17	0.49	23.53	-1182.2675
RHS (17)	2.24	-10.17	0.49	31.57	-1181.9494
RHS (18)	2.22	-10.17	0.48	31.68	-1181.9491
RHS (19)	NA	NA	NA	NA	NA
RHS (20)	2.31	-10.17	0.49	30.59	-1181.9550
RHS (21)	2.17	-10.17	0.49	31.00	-1181.9514
RHS (22)	2.07	-10.17	0.49	23.00	-1182.3078
RHS (23)	2.23	-10.17	0.49	26.95	-1182.0403
RHS (24)	2.32	-10.17	0.49	31.42	-1181.9517
RHS (25)	2.18	-10.16	0.50	20.44	-1182.7365
RHS (26)	2.22	-10.17	0.49	31.73	-1181.9491
RHS (27)	2.30	-10.17	0.49	30.78	-1181.9535
RHS (28)	2.14	-10.17	0.49	31.38	-1181.9504
RHS (29)	NA	NA	NA	NA	NA
RHS (30)	2.22	-10.17	0.49	31.73	-1181.9491

Table 11: Differences between RHS, and PSHGR Estimated Parameters for Weighted Unscaled Estimates from Simulation Run 53 from Simulation Set 11, Estimating Model from Equation 34

The next point examined is in the second panel of Figure 24, an weighted unscaled estimate of σ^2_{0k} from simulation run 53. The PSHGR and RHS results from a number of iteration points ranging from 15 to 30 are in Table 11. The table shows that the $\hat{\beta}_2$ and σ^2_ϵ are mostly unaffected by the number of iteration points. The estimates of $\hat{\beta}_0$ do vary between 1.89 and 2.32. The $\hat{\sigma}^2_{0k}$ are quite sensitive, ranging from 20.44 to 31.73. Note that there are no log likelihood values for PSHGR as there is no weighted likelihood. However, from the log likelihood values, the maximum occurs for PSHGR at iteration points 18, 26 and 30. Those corresponding estimates are close to the PSHGR estimates. When the simulations were run for simulation set 11, the first converged simulation starting with 15 iteration points was chosen. Note that increasing the number of iteration points does not produce a monotonic increase in the log likelihood, as the lowest log likelihood occurred with 25 iteration points.

8.4 Description of the MSE Results

For the misspecification of fixed effects simulation set 1, the estimating model in Equation 3 both PSHGR and RHS prefered the weighted unscaled. This is surprising, because the estimating is the correct model with no informative sampling. The unweighted estimates do not have a smaller variance than the weighted estimates in this simulation. Also the differences between the $RRMSE$'s are very small, see Section 8.5. For example, the largest PSHGR $RRMSE$ is 0.0785 and the smallest is 0.0733. For the estimated model in Equation 4 the PSHGR and RHS estimates have different weighting schemes representing the lowest $RRMSE$. The RHS methodology has the lowest $RRMSE$ for the weighted unscaled estimates. As seen in Figures 2 and 12, there are some differences between the RHS and PSHGR weighted unscaled estimates of σ_{0k}^2. This is causing the mean of the RHS method to be lower than the mean of the PSHGR method, resulting different weighting schemes producing the lowest $RRMSE$. When the estimating model is from Equation 5, the estimation of the σ_ϵ^2 is dominating the $RRMSE$ calculation. Because the weighted unscaled estimates are the smallest (i.e. closest to the true value of 0.5), both methodologies produce the smallest $RRMSE$ for the wieghted unscaled estimates. The $ARRMSE$ of PSHGR and RHS for estimated models in Equations 4 and 5 both prefer the unweighted estimates because of the smaller variances.

For the misspecification of fixed effects simulation set 4, for all the estimated models the PSHGR and RHS methods have the lowest $RRMSE$ with the unweighted estimates. Note the smaller variance from the unweighted estimators and that the weighting schemes are better at compensating for the informative sampling in the β_0, σ_{0k}^2 and σ_ϵ^2 parameters. Likely, the reason why the unweighted estimates produce the smallest $RRMSE$ is because in the σ_{0k}^2 and σ_ϵ^2 estimates, the model misspecification in Equations 8 and 9 increase the bias and the unweighted estimates are the smallest. When the model misspecification is taken into account with the $ARRMSE$, the estimated model in Equation 8 has smallest $ARRMSE$ with the weighted scaled 1 estimates. However for $ARRMSE$ in from the estimated model in Equation 9, the compensation for the bias using the weighted estimates does not overcome the smaller variance of the unweighted estimates.

For the misspecification of the random effects simulation set 5, both estimated models from Equations 11 and 12 prefer the unweighted estimates. There is no informative sampling in this simulation set and the unweighted estimates have small variances. For the estimated model in Equation 12, only the $ARRMSE$ is computed, as the true value of the σ^2_{0k} parameter is zero.

For the misspecification of the random effects simulation set 6, the estimated model in Equation 14 produces the smallest $RRMSE$ with the weighted scaled 2 estimates. In this case, the unweighted estimates are not chosen because of both bias due to the informative sampling in the β_1 and σ^2_{1k} parameters. When determining which weighting scheme produces the lowest $RRMSE$, the σ^2_{1k} parameter dominates, and the weighted unscaled 2 estimates produce the lowest $RRMSE$.

For the misspecification of the random effects simulation set 7, the unweighted estimates produce the smallest $RRMSE$ (or $ARRMSE$) all the estimated models. This is because of the smaller variance of the unweighted estimates, the lack of informative sampling, and the small variance of σ^2_{2k}.

For the misspecification of the random effects simulation set 8, the estimating model in Equation 20 produced the smallest $RRMSE$ for PSHGR and RHS with the weighted scaled 1 estimates. The informative sampling produces bias in the unweighted estimates of β_2 and σ^2_{2k}. All the weighted schemes performed well with similar $RRMSE$. The $RRMSE$ for the PSHGR weighted estimates ranged from 0.2556 to 0.2204. The estimating model in Equation 21 produced the smallest $ARRMSE$ for PSHGR and RHS with the unweighted estimates. The largest contributers to the $ARRMSE$ are the estimates of σ^2_{0k}, and the unweighted estimates have the smallest values. The small variance on the β_0 and β_1 unweighted estimates also contribute to the smaller $ARRMSE$.

For the misspecification of the stratification layering simulation set 9, the $RRMSE$

(and RAsqMSE) is lowest for the unweighted estimates for all of the estimating models. For the estimating model in Equation 23, the true value of $\sigma^2_{01k.01k}$ is zero, and the estimates from this term were not included in the MSE calculations. The terms with the largest bias are the σ^2_ϵ estimates, of which the unweighted and weighted scaled 1 estimates produce the smallest $RRMSE$. The weighted scaled 1 estimates will for RHS produce a large $RRMSE$ for the σ^2_{02k} estimates (as explained about Figure 20). The unweighted estimates have slightly smaller variances, causing them to have the smallest $RRMSE$s. For the estimating models in Equations 24 and 26, the uweighted estimates produce the smallest $ARRMSE$s due to the smaller variances and the smaller bias on the σ^2_ϵ estimates.

For the misspecification of the stratification layering simulation set 10, the estimating model in Equation 28 the smallest $RRMSE$ is with the unwieghted estimates due to the low bias and variance of the estimates. For the estimating models in Equation 29 and 31 the $RRMSE$ is the smallest with the weighted unscaled estimates. This is because the model misspecification produces large positive bias on the σ^2_ϵ parameter and the weighted unscaled estimates have the smallest value. For the estimated model in Equation 29 the unweighted estimates produced the smallest $ARRMSE$ due to the smaller variances. For the estimated model in Equation 31, PSHGR and RHS produced different results. Notice that the PSHGR weighted scaled 1 estimates of σ^2_ϵ have a low 0.025 quantile, as seen in Figure 9, 24 and 26. The weighted scaled 1 estimates produced the lowest $ARRMSE$ for the RHS method and the weighted scaled 2 estimates produced the lowest $ARRMSE$ for the PSHGR method.

For the misspecification of the clustering layers, simulation set 11, the estimated model in Equation 33 contains no model misspecification. As expected, the $RRMSE$ for PSHGR is lowest for the unweighted estimates due to the minimal bias and smaller variance. However, the $RRMSE$ for RHS is lowest for the the weighted scaled 1 estimates. This is due to the very large bias in the unweighted estimate of σ^2_{02k}. The RHS weighted scaled 1 estimate of σ^2_{0k} is better behaved and generally has a smaller variance than the weighted scaled 2 estimates. For the estimated model in Equation 34, the $ARRMSE$ for both PSHGR and RHS favor the unweighted estimates due to the lower variance and the lack of informative sampling bias. For the estimating model in Equation 35, the $RRMSE$ favors the weighted unscaled estimates, as the $RRMSE$ is dominated by the σ_ϵ^2 term and the weighted unscaled estimates are closest to the true value. When adjusting it for the anticipated values, the $ARRMSE$ for both RHS and PSHGR favor the unweighted estimates due ot the low variance and the lack of model misspecification bias. Finally, for the estimated model in Equation 36, the $ARRMSE$ favors the unweighted estimates due to the smaller variance and the lack of informative sampling bias.

For the misspecification of the clustering layering simulation set 12, both estimating models contain model misspecification. For the estimated model in Equation 38, the $RRMSE$ is dominated by the bias in the σ_ϵ^2 estimates and the weighted unscaled estimates have the lowest mean. For the RAsqMSE, the unweighted estimates produce the lowest numbers because of the low variance and minimal bias. For the estimated model in Equation 39, the $RRMSE$ is domiated by the bias in the σ_{0k1k2}^1 estimates. The weighted scaled 1 estimates have the lowest $RRMSE$ for the σ_{0k10k2}^2 parameter, so they also produce the lowest $RRMSE$ for the estimated model.

Tables 12 and 13 contain the numeric values of the $RRMSE$ and $ARRMSE$ for each simulation.

	Eqn. Num.	Weighting Scheme with Lowest MSE							
		Unweighted		Weighted Unscaled		Weighted Scaled 1		Weighted Scaled 2	
		RRMSE		*RRMSE*		*RRMSE*		*RRMSE*	
		P	R	P	R	P	R	P	R
Mis Fix 1	3	7.8594e-2	7.8587e-2	**7.3379e-2**	**7.3410e-2**	7.77096e-2	7.7317e-2	7.7096e-2	7.7317e-2
	4	3.2644e+4	2.8374e+4	3.1237e+4	**2.7002e+4**	**3.1143e+4**	3.2019e+4	3.1218e+4	2.9339e+4
	5	4.0222e+4	3.9975e+4	**3.1438e+4**	**3.1609e+4**	4.5315e+4	3.9533e+4	3.5561e+4	3.5635e+4
Mis Fix 4	7	**1.1561e-1**	**1.1791e-1**	2.0433e-1	2.0578e-1	2.0171e-1	2.0479e-1	1.8831e-1	1.9138e-1
	8	**1.4261e+4**	**1.3857e+4**	2.5140e+4	2.3402e+4	2.5084e+4	2.5006e+4	2.5119e+4	2.3461e+4
	9	**1.6398e+4**	**1.6261e+4**	2.6131e+4	2.6022e+4	1.8482e+4	1.8546e+4	2.3537e+4	2.3488e+4
Mis Ran 5	11	**9.4797e-2**	**9.4884e-2**	2.4933e-1	2.5020e-1	2.3252e-1	2.3242e-1	2.2132e-1	2.1994e-1
	12	—	—	—	—	—	—	—	—
Mis Ran 6	14	2.5834e-1	2.5846e-1	2.3064e-1	2.3085e-1	2.1135e-1	2.1013e-1	**2.0382e-1**	**2.0223e-1**
	15	—	—	—	—	—	—	—	—
Mis Ran 7	17	**9.5798e-2**	**9.7300e-2**	2.2984e-1	2.3196e-1	2.0004e-1	2.0086e-1	2.0581e-1	2.0591e-1
	18	—	—	—	—	—	—	—	—
Mis Ran 8	20	3.1733e-1	3.1731e-1	2.5557e-1	2.5613e-1	**2.2042e-1**	**2.2037e-1**	2.2716e-1	2.2711e-1
	21	—	—	—	—	—	—	—	—
Mis Strat 9	23	**6.8850e-1**	**6.8652e-1**	9.1607e-1	9.5862e-1	8.8713e-1	3.8200e+5	8.9394e-1	8.8134e-1
	24	—	—	—	—	—	—	—	—
	26	—	—	—	—	—	—	—	—
Mis Strat 10	28	**5.7122e-1**	**5.6701e-1**	7.3152e-1	7.3170e-1	6.9078e-1	6.901e-1	7.0105e-1	6.9815e-1
	29	—	—	—	—	—	—	—	—
	31	—	—	—	—	—	—	—	—
Mis Strat 11	33	**7.3889e-1**	1.1001e+4	8.4443e-1	4.5244e+4	8.2915e-1	**1.7074**	8.2344e-1	3.4923e+6
	34	—	—	—	—	—	—	—	—
	35	2.6340e+3	2.6309e+3	**2.0145e+3**	**2.0547e+3**	2.4009e+3	2.5901e+3	2.2253e+3	2.0565e+10
	36	—	—	—	—	—	—	—	—
Mis Clust 12	38	3.6977	3.6990	**2.4296**	**2.4310**	3.7148	3.6647	2.8694	2.9982
	39	1.7618e+1	1.7670e+1	**1.4982e+1**	**1.4745e+1**	1.5355e+1	1.5391e+1	1.5167e+1	1.4876e+1

Table 12: Relative Root Mean Square Error (*RRMSE*) for each Simulation Set

	Eqn. Num.	Unweighted		Weighted Unscaled		Weighted Scaled 1		Weighted Scaled 2	
		ARRMSE		*ARRMSE*		*ARRMSE*		*ARRMSE*	
		P	R	P	R	P	R	P	R
Mis Fix 1	3	...	...	**...**	**...**	...	...	...	...
	4	**8.4129e-2**	**1.0717e-1**	2.7580e-1	2.8260e-1	2.4200e-1	2.4138e-1	2.5024e-1	2.5996e-1
	5	**1.6410e+1**	**1.4602e+1**	3.5597e+3	3.7144e+3	1.9264e+2	5.7353e+1	2.0935e+3	2.1438e+3
Mis Fix 4	7	...	...	**...**	**...**	...	...	...	...
	8	6.7070e-1	6.7879e-1	3.7168e-1	3.8074e-1	**3.5823e-1**	**3.5876e-1**	3.6103e-1	3.6468e-1
	9	**8.1960e+1**	**7.8062e-1**	4.6589e+3	4.6490e+3	4.6319e+2	4.6718e+2	3.2840e+3	3.4033e+3
Mis Ran 5	11	...	...	**...**	**...**	...	...	...	...
	12	**5.8642e-1**	**5.9620e-1**	1.3494	1.3550	1.3171	1.3100	1.3242	1.3283
Mis Ran 6	14	...	...	**...**	**...**	...	...	...	...
	15	**7.6845e-1**	**7.6157e-1**	1.4761	1.3475	1.4331	1.4335	1.4456	1.4437
Mis Ran 7	17	...	...	**...**	**...**	...	...	...	...
	18	**6.8822e-1**	**7.1810e-1**	2.1469	2.1804	1.4199	1.4362	1.7865	1.8101
Mis Ran 8	20	...	...	**...**	**...**	...	...	...	...
	21	**7.5426e-1**	**7.6435e-1**	1.0430	1.0620	9.5318e-1	9.6936e-1	8.9989e-1	9.2430e-1
Mis Strat 9	23	...	...	**...**	**...**	...	...	...	...
	24	**4.1869e-1**	**4.1724e-1**	2.7596	2.8129	2.7142	2.7195	2.7312	2.7314
	26	**1.6187**	**1.6190**	1.7263	1.7411	1.6772	1.6801	1.6952	1.7001
Mis Strat 10	28	...	...	**...**	**...**	...	...	...	...
	29	**1.3675**	**1.3722**	2.6450	2.6218	2.7750	2.5768	2.5962	2.6028
	31	3.1449	3.1353	2.7032	2.7108	2.8265	**2.6567**	**2.6964**	2.7327
Mis Strat 11	33	...	...	**...**	**...**	...	...	...	...
	34	**1.1087**	**1.0060**	1.2856	1.1450	1.2356	1.2328	1.2602	1.1857
	35	**1.2517**	**1.4311**	7.1681	1.3923e+1	2.6557	2.1000	5.7717	2.0565e+10
	36	**1.6316**	**1.6278**	4.3153	4.3419	4.3551	4.3528	4.3217	4.3244
Mis Clust 12	38	**3.8637e-1**	**3.8616e-1**	5.2995e-1	5.1567e-1	4.7571e-1	4.4115e-1	4.5929e-1	4.5142e-1
	39	3.6999	3.7008	**2.4401**	**2.4438**	3.7290	3.6796	2.8812	3.0110

Weighting Scheme with Lowest MSE

Table 13: Anticipated Relative Root Mean Square Error (*ARRMSE*) for each Simulation Set

8.5 Tables of True and Anticipated Parameter Values

For the computation of the $ARRMSE$ values, the anticipated parameter values are needed. The derivation of these values is in Section 8.1. They are also included in Tables 14 and 15 for reference.

	Eqn. Num.	True (Anticipated) Parameter Values β_0	β_1	β_2	σ^2_{0k}	σ^2_{ϵ}	σ^2_{1k}	σ^2_{2k}
Mis Fix 1	3	1	-2	2	0.2	0.5		
	4	1(-5)	-2 (NA)	2 (2)	0.2 (36.2)	0.5 (0.5)		
	5	1(3)	-2 (-2)	2 (NA)	0.2 (0.2)	0.5 (100.5)		
Mis Fix 4	7	1	-2	2	0.2	0.5		
	8	1(-5)	-2 (NA)	2 (2)	0.2 (36.2)	0.5 (0.5)		
	9	1(3)	-2 (-2)	2 (NA)	0.2 (0.2)	0.5 (100.5)		
Mis Ran 5	11	1	-2	2		0.5	1	
	12	1(1)	-2 (-2)	2 (2)	0 (18)	0.5 (0.5)	1(NA)	
Mis Ran 6	14	1	-2	2		0.5	1	
	15	1(1)	-2 (-2)	2 (2)	0 (18)	0.5 (0.5)	1(NA)	
Mis Ran 7	17	1	-2	2		0.5		0.8
	18	1(1)	-2 (-2)	2 (2)	0 (1.6)	0.5 (16.5)		0.8(NA)
Mis Ran 8	20	1	-2	2		0.5		0.8
	21	1(1)	-2 (-2)	2 (2)	0 (1.6)	0.5 (16.5)		0.8(NA)

Table 14: True and Anticipated Parameter Values for Simulation Sets 1-8.

	Eqn. Num.	True (Anticipated) Parameter Values β_0	β_1	σ^2_{01k}	σ^2_{02k}	$\sigma^2_{01k.02k}$ [13]	σ^2_{ϵ}	σ^2_{0k} or $\sigma^2_{0k_1}$	$\sigma^2_{0k_1k_2}$ or β_2
Mis Strat 9	23	-3	8	1	5	0	0.5		
	24	-3(1)	8 (NA)	1 (NA)	1 (NA)	0(NA)	0.5 (0.5)	0 (16)	
	26	-3(1)	8 (NA)	1 (NA)	1 (NA)	0 (NA)	0.5 (0.5)	0 (16)	
Mis Strat 10	28	-3	8				0.5	5	
	29	-3 (1)	8 (NA)				0.5 (16.5)	5 (5)	
	31	-3 (1)	8 (NA)				0.5 (16.5)	5 (5)	
Mis Strat 11	33	7	-8	1	5	0	0.5		-10
	34	7 (3)	-8 (NA)	1 (NA)	5 (NA)	0 (NA)	0.5 (NA)	0 (0)	-10 (-10)
	35	7 (2)	-8 (-8)	1 (1)	5 (5)	0 (0)	0.5 (NA)		-10 (NA)
	36	7 (-1)	-8 (NA)	1 (NA)	5 (NA)	0 (NA)	0.5 (25.5)	0 (16.5)	-10 (NA)
Mis Clust 12	38	5 (5)					0.5 (1.5)	5 (5)	1 (NA)
	39	5 (5)					0.5 (0.5)	5 (NA)	1 (6)

Table 15: True and Anticipated Parameter Values for Simulation Sets 9-12.

8.6 Computer Code

The simulations for PSHGR method were run using c-code I developed. This code may be found at `http://stat.cmu.edu` under the **Recent PhD Theses** link. The c-code uses the VMR library, downloaded from `http://www.stat.cmu.edu/~hseltman/`. It is in the Computer Programming, C/C++ section. The code uses `blas` functions, downloadable from `http://www.netlib.org`. The compilation instructions are commented in the beginning of the code. Along with the code are sample input files and the corresponding output file.

The simulations for the RHS method were run in `stata` using the `gllamm()` routine. The `gllamm()` routine can be found at `http://www.gllamm.org`.

References

Asparouhov, T. (2006). General multi-level modeling with sampling weights. *Communications in Statistics – Theory and Methods*, 35:439 – 460.

Fienberg, S. E. (1989). Modeling considerations: Discussion from a modeling perspective. In Kasprzyk, D., Duncan, G., Kalton, G., and Singh, M. P., editors, *Panel Surveys*, pages 512–539. Wiley.

Hoem, J. (1989). The issue of weights in panel surveys of individual behavior. In Kasprzyk, D., Duncan, G., Kalton, G., and Singh, M. P., editors, *Panel Surveys*, pages 512–539. Wiley.

Huang, R. and Hidiroglou, M. (2003). Design consistent estimators for a mixed linear model on survey data. In *Proceedings of the Survey Research Methods Section, American Statistical Association (2003)*, pages 1897–1904.

Kalton, G. (1989). Modeling considerations: Discussion from a survey sampling perspective. In Kasprzyk, D., Duncan, G., Kalton, G., and Singh, M., editors, *Panel Surveys*, pages 575–585. Wiley.

Korn, E. L. and Graubard, B. I. (2003). Estimating variance components by using survey data. *Journal of the Royal Statistical Society, Series B*, 65(1):175 – 190.

Little, R. (2004). To model or not to model? competing modes of inference for finite population sampling. *Journal of the Americal Statistical Association*, 99:546–556.

Little, R. and Rubin, D. (2002). *Statistical Analysis with Missing Data*. Wiley.

Mislevy, R. J. and Sheehan, K. M. (1989). The role of collateral information about examinees in item parameter estimation. *Psychometrika*, 54(4):661–679.

Patterson, B., Dayton, M., and Graubard, B. (2002). Latent class analysis of complex sample data: Application to dietary data. *Journal of the American Statistical Association*, 97(459):1–21.

Pfeffermann, D., Skinner, C., Holmes, D., Goldstein, H., and Rasbash, J. (1998). Weighting for unequal selection probabilities in multilevel models. *Journal of the Royal Statistical Society, Series B*, 60(1):23 – 40.

Rabe-Hesketh, S. and Skrondal, A. (2006). Multilevel modelling of complex survey data. *Journal of the Royal Statistical Society: Series A*, 169:805–827.

Searle, S., Casella, G., and McCulloch, C. (1992). *Variance Components*. John Wiley & Sons, Inc.

Stapleton, L. M. (2002). The incorporation of sample weights into multilevel structural equation models. *Structural Equation Modeling*, 9(4):475–502.

Thomas, D. R. and Cyr, A. (2002). Applying item response theory methods to complex survey data. In *Proceedings of the Survey Methods Section*, pages 17–25. Statistical Society of Canada.