7.1 Measure on Product Spaces

7.1.1 Measurable product spaces

Definition 7.1 (Product $\sigma$-Field) Let $(\Omega_1, F_1), (\Omega_2, F_2)$ be two measurable spaces. The product $\sigma$-field $F_1 \otimes F_2$ on $\Omega_1 \times \Omega_2$ is defined as the $\sigma$-field generated by the collection of all sets of the form $\{A_1 \times A_2 : A_1 \in F_1, A_2 \in F_2\}$. The sets in this collection are called measurable rectangles.

Remark 7.2 $F_1 \otimes F_2 \neq F_1 \times F_2$ because $F_1 \times F_2$ may not be closed on $A^c$ or $A_1 \cup A_2$. (Consider $(\mathbb{R}^2, B^2)$.)

Remark 7.3 The collection of measurable rectangles is a $\pi$-system.

Definition 7.4 (Coordinate Projection) For $i = 1, 2$ the coordinate projection $\pi_i : \Omega_1 \times \Omega_2 \mapsto \Omega_i$ is defined as $\pi_i(\omega_1, \omega_2) = \omega_i$.

Claim 7.5 $F_1 \otimes F_2$ is the smallest $\sigma$-field such that the coordinate projections are all measurable.

Claim 7.6 The $k$ dimensional Borel $\sigma$-field satisfies $B^k = B^1 \otimes \ldots \otimes B^1$.

Proposition 7.7 (Properties) Let $(\Omega_1, F_1), (\Omega_2, F_2)$ be two measurable spaces:

- For each $B \in F_1 \otimes F_2$ and each $\omega_1 \in \Omega_1$ the $\omega_1$-section of $B$, $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B\}$ is in $F_2$.
- If $\mu_2$ is a $\sigma$-field on $(\Omega_2, F_2)$ then $\forall B \in F_1 \otimes F_2$ the function $f : \Omega_1 \mapsto \mathbb{R}$ defined by $f(\omega_1) = \mu_2(B_{\omega_1})$ is measurable.
- If $f : \Omega_1 \times \Omega_2 \mapsto (S, A)$ is measurable then $\forall \omega_1 \in \Omega_1$ the function $f_{\omega_1} : \Omega_2 \mapsto S$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$ is measurable.
- If $\mu_2$ is $\sigma$-finite on $(\Omega_2, F_2)$ and $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ be measurable and nonnegative then the function $g : \Omega_1 \mapsto \mathbb{R}^{0+}$ defined by $g(\omega_1) = \int f(\omega_1, \omega_2) \, d\mu_2(\omega_2)$ is measurable.
Proof of the first property:

**Proof:** Fix \( \omega_1 \in \Omega_1 \). Let \( C_{\omega_1} = \{ B \in \mathcal{F}_1 \otimes \mathcal{F}_2 : B_{\omega_1} \in \mathcal{F}_2 \} \). First show that \( C_{\omega_1} \) is a \( \sigma \)-field:

- For \( B = \Omega_1 \times \Omega_2, B_{\omega_1} = \{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \} = \Omega_2 \in \mathcal{F}_2 \).
- For \( B \in C_{\omega_1} \), we have \( B_{\omega_1} \in \mathcal{F}_2 \) thus \( B_{\omega_1} \in \mathcal{F}_2 \). Consider \( (B^c)_{\omega_1} = \{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B^c \} \). Recall that \( B_{\omega_1} = \{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B \} \). We have \( (B^c)_{\omega_1} \cap B_{\omega_1} = \emptyset \) and \( (B^c)_{\omega_1} \cup B_{\omega_1} = \Omega_2 \). Hence \( (B^c)_{\omega_1} = B^c_{\omega_1} \) and \( (B^c)_{\omega_1} \in \mathcal{F}_2 \). Since \( B \in C_{\omega_1} \subset \mathcal{F}_1 \otimes \mathcal{F}_2 \) we have \( B^c \in \mathcal{F}_1 \otimes \mathcal{F}_2 \). Therefore, \( B^c \in C_{\omega_1} \).
- Consider \( B = \bigcup_{n=1}^{\infty} B_n \) where \( B_n \in C_{\omega_1} \), for all \( n \), i.e. \( (B_n)_{\omega_1} \in \mathcal{F}_2 \) for all \( n \). We have \( B_n \in \mathcal{F}_1 \otimes \mathcal{F}_2 \) for all \( n \) so \( B \in \mathcal{F}_1 \otimes \mathcal{F}_2 \). We will show \( B_{\omega_1} = \{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \bigcup_{n=1}^{\infty} B_n \} = \bigcup_{n=1}^{\infty} (B_n)_{\omega_1} \): For any \( \omega_2 \in B_{\omega_1} \) there exists \( n \) such that \( (\omega_1, \omega_2) \in B_n \). Hence \( \omega_2 \in (B_n)_{\omega_1} \) and \( \omega_2 \in \bigcup_{n=1}^{\infty} (B_n)_{\omega_1} \).

Now we have shown that \( B_{\omega_1} = \bigcup_{n=1}^{\infty} (B_n)_{\omega_1} \in \mathcal{F}_2 \) which indicates that \( B \in C_{\omega_1} \) holds.

Therefore \( C_{\omega_1} \) is a \( \sigma \)-field.

Now consider \( A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2 \). Then we have \( (A_1 \times A_2)_{\omega_1} = A_2 \) if \( \omega_1 \in A_1 \) and \( (A_1 \times A_2)_{\omega_1} = \emptyset \) if \( \omega_1 \notin A_1 \). In any case we have \( (A_1 \times A_2)_{\omega_1} \in \mathcal{F}_2 \) and thus \( A_1 \times A_2 \in C_{\omega_1} \). Therefore, all measurable rectangles in the form of \( A_1 \times A_2 \) are contained in \( C_{\omega_1} \). According to the fact that \( C_{\omega_1} \) is a \( \sigma \)-field and the definition of \( \mathcal{F}_1 \otimes \mathcal{F}_2 \) we have \( C_{\omega_1} \subset \mathcal{F}_1 \otimes \mathcal{F}_2 \). We also have \( C_{\omega_1} \subset \mathcal{F}_1 \otimes \mathcal{F}_2 \) by definition. Therefore \( C_{\omega_1} = \mathcal{F}_1 \otimes \mathcal{F}_2 \), which means, for all \( B \in \mathcal{F}_1 \otimes \mathcal{F}_2, B_{\omega_1} \in \mathcal{F}_2 \) holds.

**Lemma 7.8** Let \((\Omega_1, \mathcal{F}_1), (\Omega_2, \mathcal{F}_2), (S_1, A_1)\) and \((S_2, A_2)\) be measurable spaces. For \( i = 1, 2 \) let \( f_i : \Omega_i \rightarrow S_i \) be a function. Define function \( g : \Omega_1 \times \Omega_2 \rightarrow S_1 \times S_2 \) by \( g(w_1, w_2) = (f_1(\omega_1), f_2(\omega_2)) \). Then \( g \) is \( \mathcal{F}_1 \otimes \mathcal{F}_2/A_1 \otimes A_2 \) measurable if and only if \( f_i \) is \( \mathcal{F}_i/A_i \) measurable for \( i = 1, 2 \).

### 7.1.2 Product measures

**Theorem 7.9 (Product measure)** Let \((\Omega_1, \mathcal{F}_1, \mu_1)\) and \((\Omega_2, \mathcal{F}_2, \mu_2)\) be two measurable spaces where \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-finite measures. There exists a unique measure \( \mu \) on \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\) that satisfies \( \mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2) \) for all \( A_1 \in \mathcal{F}_1 \) and \( A_2 \in \mathcal{F}_2 \). This measure is called the product measure, written as \( \mu = \mu_1 \times \mu_2 \).

**Proof:**

**Uniqueness:**

First we show that any such measure must be \( \sigma \)-finite. Since \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-finite there exist \( \{ A_n \}_{n=1}^{\infty} \in \mathcal{F}_1 \) and \( \{ B_n \}_{n=1}^{\infty} \in \mathcal{F}_2 \) such that \( \bigcup_{n=1}^{\infty} A_n = \Omega_1, \bigcup_{n=1}^{\infty} B_n = \Omega_2, \mu_1(A_n) \) and \( \mu_2(B_n) \) are finite for all \( n \). Consider \( \bigcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j \). For and \( (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \) there exists \( i, j \) such that \( \omega_1 \in A_i \) and \( \omega_2 \in B_j \), which means \( (\omega_1, \omega_2) \in A_i \times B_j \). Hence \( \bigcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j = \Omega_1 \times \Omega_2 \). For any \( (i,j) \in \mathbb{N}^2 \) we have \( \mu(A_i \times B_j) = \mu_1(A_i) \mu_2(B_j) < \infty \). Since \( \mathbb{N}^2 \) is a countable set we can conclude that \( \mu \) is \( \sigma \)-finite.

Suppose there are two measures \( \mu \) and \( \mu' \) satisfying the condition in the theorem. Recall that the collection of measurable rectangles \( \{ A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2 \} \) is a \( \pi \)-system. \( \mu \) and \( \mu' \) are both \( \sigma \)-finite and agree
on this π-system. By Uniqueness theorem they agree on the generated σ-field \( \mathcal{F}_1 \otimes \mathcal{F}_2 \), i.e., \( \mu = \mu' \), which means such measure must be unique.

**Existence:**

For any \( B \in \mathcal{F}_1 \otimes \mathcal{F}_2 \) let \( \mu(B) = \int_{\Omega_1} \mu_2(B_{\omega_1})d\mu_1(\omega_1) \) where \( B_{\omega_1} = \{ \omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \} \) as introduced previously. Then \( \mu \) is a measure.

For any \( A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2 \),

\[
\mu(A_1 \times A_2) = \int_{\Omega_1} \mu_2((A_1 \times A_2)_{\omega_1})d\mu_1(\omega_1) = \int_{\Omega_1} \mathbb{I}_{A_1} \mu_2(A_2)d\mu_1(\omega_1) = \mu_2(A_2) \int_{\Omega_1} \mathbb{I}_{A_1} d\mu_1(\omega_1) = \mu_1(A_1) \mu_2(A_2). 
\]

Hence such measure exists.

**Theorem 7.10 (Tonelli/Fubini theorem)** Let \((\Omega_1, \mathcal{F}_1, \mu_1)\) and \((\Omega_2, \mathcal{F}_2, \mu_2)\) be two measurable spaces where \( \mu_1 \) and \( \mu_2 \) are σ-finite measures. Let \( \mu = \mu_1 \times \mu_2 \) be the product measure on \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\). Let \( f : \Omega_1 \times \Omega_2 \to \mathbb{R} \) be a nonnegative measurable function. (Can be extended to integrable functions with respect to the product measure \( \mu \), i.e. \( \int |f|d\mu < \infty \).) Then the following holds:

\[
\int f d\mu = \int \left[ \int f(\omega_1, \omega_2) d\mu_1(\omega_1) \right] d\mu_2(\omega_2) = \int \left[ \int f(\omega_1, \omega_2) d\mu_2(\omega_2) \right] d\mu_1(\omega_1).
\]

### 7.2 Independence

**Definition 7.11 (Independence between collection of sets)** Let \((\Omega, \mathcal{F}, P)\) be a probability space. For two collections \( \mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{F} \), we say that \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are independent if \( P(A_1 \cap A_2) = P(A_1)P(A_2) \) for all \( A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2 \).

**Definition 7.12 (Independence between random variables)** Let \((\Omega, \mathcal{F}, P)\) be a probability space. For \( i = 1, 2 \) let \((S_i, \mathcal{A}_i)\) be measurable spaces and \( X_i : \Omega \to S_i \) be \( \mathcal{F}/\mathcal{A}_i \) measurable functions. (Hence \( X_1 \) and \( X_2 \) are random variables.) Let \( \sigma(X_i) \) be the σ-field \( X_i^{-1}(\mathcal{A}_i) \subset \mathcal{F} \) generated by function \( X_i \). We say that \( X_1 \) and \( X_2 \) are independent if \( \sigma(X_1) \) and \( \sigma(X_2) \) are independent collections.

**Theorem 7.13** Let \( X_1, X_2 \) be two random variables following the definition above. Define another random variable \( X : \Omega \to S_1 \times S_2 \) by \( X = (X_1, X_2) \). Then its distribution \( \mu_X \) (induced measure on \((S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2)\)) is the product measure \( \mu_{X_1} \times \mu_{X_2} \) if and only if \( X_1 \) and \( X_2 \) are independent.

**Proof:** By definition \( X_1 \) and \( X_2 \) are independent if and only if for all \( B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2) \) we have \( P(B_1 \cap B_2) = P(B_1)P(B_2) \). It remains to show that \( \mu_X = \mu_{X_1} \times \mu_{X_2} \) if and only if \( \forall B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2), P(B_1 \cap B_2) = P(B_1)P(B_2) \).

**Proof of if.** Suppose \( \forall B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2), P(B_1 \cap B_2) = P(B_1)P(B_2) \) holds. For any \( A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, \) we have

\[
\mu_X(A_1 \times A_2) = P(\{\omega \in \Omega : X_1(\omega) \in A_1, X_2(\omega) \in A_2\}) = P(X_1^{-1}(A_1) \cap X_2^{-1}(A_2)) \\
= P(X_1^{-1}(A_1))P(X_2^{-1}(A_2)) = \mu_{X_1}(A_1)\mu_{X_2}(A_2).
\]

Therefore, \( \mu_X = \mu_{X_1} \times \mu_{X_2} \).
Proof of only if. Suppose \( \mu_X = \mu_{X_1} \times \mu_{X_2} \). Then for all \( B_1 \in X_1^{-1}(A_1), B_2 \in X_2^{-1}(A_2) \),
\[
P(B_1 \cap B_2) = P(X_1^{-1}(X_1(B_1)) \cap X_2^{-1}(X_2(B_2))) = P(X^{-1}(X_1(B_1) \times X_2(B_2)))
\]
\[
= \mu_X(X_1(B_1) \times X_2(B_2)) = \mu_{X_1}(X_1(B_1))\mu_{X_2}(X_2(B_2)) = P(B_1)P(B_2).
\]

\[\square\]

7.3 Stochastic Processes

Definition 7.14 Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(T\) be a set. For each \(t \in T\), there is a measurable space \((X_t, \mathcal{F}_t)\) and a random variable \(X_t : \Omega \mapsto X_t\). The collection \(\{X_t : t \in T\}\) is called a stochastic process, and \(T\) is called the index set.

Example 7.15 Let \(T = \{1, ..., k\}\). A vector of random variables \(X = [X_1, ..., X_k]\) is a stochastic process.

Example 7.16 (Random probability measure) Let \(\Theta : \Omega \mapsto \mathbb{R}\) be a random variable, \(f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}\) be a nonnegative function such that \(\int_{\mathbb{R}} f(x, \theta) \, dx = 1\) for all \(\theta \in \mathbb{R}\). For example, \(f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(x-\theta)^2}{2}\right)\).

Let \(T = B\). For each \(B \in \mathcal{B}\) consider random variable \(X_B : \Omega \mapsto \mathbb{R}\) defined by \(X_B(\omega) = \int_B f(x, \Theta(\omega)) \, dx\). Then the stochastic process \(\{X_B : B \in \mathcal{B}\}\) is a random probability measure.

Example 7.17 (Empirical measure) Let \(X_1, ..., X_n\) be i.i.d. samples from some \(P\) on \(\mathbb{R}\). Define the empirical measure \(P_n\) on \((\mathbb{R}, \mathcal{B})\) as \(P_n(B) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \in B\}\) for all \(B \in \mathcal{B}\). (Why introduced here?)

Remark 7.18 The product set \(X = \prod_{t \in T} X_t\) can be viewed as the set of all functions \(f : T \mapsto \bigcup_{t \in T} X_t\) such that \(f(t) \in X_t\) for all \(t \in T\). For example, when \(X_t = \mathcal{Y}\) for all \(t\), \(X = \prod_{t \in T} X_t = \mathcal{Y}^T\) is the set of all functions from \(T\) to \(\mathcal{Y}\). In a stochastic process, the random variable \(X : \Omega \mapsto X\) defined by \(X(\omega) = \{X_t(\omega) : t \in T\}\) induces a probability distribution over \(X = \prod_{t \in T} X_t\), i.e. over all functions \(f : T \mapsto \bigcup_{t \in T} X_t\) such that \(f(t) \in X_t\) for all \(t \in T\).