A characterization of monotone unidimensional latent variable models *

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The Ideas

Motivation

Typical psychometrics / latent variable / errors-in-variables setting:

For repeated measures $\underline{X} = (X_1, X_2, \ldots)$, the "standard models" assume a *monotone unidimensional representation*,

Conditional Independence (CI): $\coprod \underline{X}|\Theta$ *(i. but not i.d.!)*.

Monotonicity (M): $P[X_i > t|\theta]$ nondecreasing in θ, for all t.

Unidimensionality ($d_L = 1$): $\Theta \in \mathbb{R}$.

Examples:

- *Errors in Variables, "True Score Theory", or One-Factor Factor Analysis:*

$$X_i = \alpha_i T + \epsilon_i;\ i = 1, 2, \ldots;\ \alpha_i \geq 0$$

- *"Item Response Models", e.g. used in Educational Testing:*

$$P(x_1, \ldots, x_J) = \int \prod_{j=1}^{J} P_j(\theta)^{x_j}[1 - P_j(\theta)]^{1-x_j}\, dF(\theta)$$

  where $(X_1, X_2, \ldots)$ are binary and each $P_j(\theta)$ is nondecr. in θ.

Example to think about

- The *generalized beta-binomial* model:

$$P(x_1, \ldots, x_J) = \int u^{x+} (1-u)^{J-x+} \, dG(u)$$

$$= \mathbf{G}(x_+)$$

- The *Rasch* model:

$$P(x_1, \ldots, x_J) = \int \prod_{j=1}^{J} \frac{e^{x_j(\theta-\beta_j)}}{1+e^{x_j(\theta-\beta_j)}} \, dF(\theta)$$

$$= e^{-\sum_j \beta_j x_j} \int u^{x+} \prod_{j=1}^{J} (1 + u\, e^{-\beta_j})^{-1} \, dH(u)$$

$$= e^{-\sum_j \beta_j x_j} \, \mathbf{H}(x_+)$$

What's the difference?

$(X_1, X_2, \ldots)$ are **exchangeable** iff for every J and every permutation

$$i(1),\ i(2),\ \ldots,\ i(J)$$

the distribution of

$$(X_1,\ X_2,\ \ldots,\ X_J) \quad \text{and} \quad (X_{i(1)}, X_{i(2)}, \ldots, X_{i(J)})$$

are the same.

Problem

We only get to see $\underline{X} = (X_1, X_2, \ldots)$, and must guess what model, or class of models, makes sense.

Goal:

- What features does a ***monotone unidimensional representation*** imply about the joint distribution of $\underline{X}$?
- What features of the distribution of $\underline{X}$ imply a ***monotone unidimensional representation***?

Analogue:

De Finetti, exchangeability, and the tail σ-field:

- $P(X_1, \ldots, X_J) = \int u^{x_+}(1-u)^{J-x_+} dG(u)$, $\forall J$, iff $(X_1, X_2, \ldots)$ [binary] are ***exchangeable***; in this case also $\sigma(\Theta) = \tau(\underline{X})$, the ***tail*** $\boldsymbol{\sigma}$***-field***, a.s. (e.g. Galambos, 1982).
- Aldous (1981): Any exchangeable sequence may be viewed as i.i.d. disturbances of a single underlying Θ.
- Diaconis and Freedman (1984): "Exchangeable-like" extreme-point representations by specifying sufficient statistics and conditional laws of the data given the sufficient statistics.

What property of the sequence $(X_1, X_2, \ldots)$ replaces ***exchangeability*** as a criterion for CI ***(but not i.d.)*** and M?

Observable constraints

What are the *constraints* on the the repeated measures $(X_1, X_2, \ldots)$ when the *monotone unidimensional representation* holds?

Conditional Association (CA):

Holland and Rosenbaum (1986) show that the monotone unidimensional representation implies CA: for all J, all partitions $(X_1, \ldots, X_J) = (\underline{Y}, \underline{Z})$, all nondecreasing $f(\cdots)$ and $g(\cdots)$, and all $\underline{h}(\cdots)$,

$$\text{Cov}\,(f(\underline{Y}), g(\underline{Y})|\underline{h}(\underline{Z}) = \underline{c}) \geq 0$$

- If each X_i is driven monotonically by the same Θ, then $(X_1, X_2, \ldots)$ possesses so much internal coherence that all monotone summaries of $\underline{Y}$ should be correlated, in any reasonable data-defined subpopulation.

Vanishing Conditional Dependence (VCD):

Suppose the monotone unidimensional representation holds, with $\Theta \in \tau(\underline{X})$. Then by standard approximation arguments, for all partitions $(X_1, \ldots, X_J) = (\underline{Y}, \underline{Z})$, and all $f(\cdots)$ and $g(\cdots)$,

$$\lim_{m\to\infty} \text{Cov}\,(f(\underline{Y}), g(\underline{Z})|X_{J+1}, \ldots, X_{J+m}) = 0$$

- If $\Theta \in \tau(\underline{X})$, then conditioning on $(X_{J+1}, \ldots, X_{J+m})$ approximates conditioning on Θ, for which independence holds.

Informal Definitions

- A *sigma field* $\sigma(X_1, \ldots, X_J)$ is the set of "all" inferences, parameters, and functions of $X_1, \ldots, X_J$ that can be known from observing $X_1, \ldots, X_J$.

- The *tail sigma field* of $\underline{X} = (X_1, X_2, \ldots)$,

$$\tau(\underline{X}) = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, X_{n+2}, \ldots),$$

  is the set of "all" inferences, parameters and functions that can be consistently estimated from $X_1, X_2, \ldots$, *even if some (indeed, <u>any</u>!) finite set of X_j's are ignored.*

<u>*Junker (1993):*</u>

- Argues that $\Theta \in \tau(\underline{X})$ is a *requirement* for latent variable models that are "useful" for measurement.

- Proves that for binary data, the monotone unidimensional ($d_L = 1$) IRT model does satisfies the stronger condition $\sigma(\Theta) = \tau(\underline{X})$.

<u>The Theorem</u>

1. The following are equivalent:
   (a) There exists $\Theta \in \tau(\underline{X})$ such that the *monotone unidimensional representation* holds.
   (b) There exists a σ-field $\mathcal{F} \subseteq \tau(\underline{X})$ such that
      i. $\coprod \underline{X}|\mathcal{F}$; and
      ii. For all J, all nondecreasing $f(\cdots)$ and $g(\cdots)$, and all $A \in \mathcal{F}$ for which the covariance below is defined,
      $$\mathrm{Cov}\,(f(X_1,\ldots,X_J), g(X_1,\ldots,X_J)|A) \geq 0. \qquad (*)$$
   (c) CA and VCD hold for $\underline{X} = (X_1, X_2, \ldots)$.
2. When any (hence all) of the above conditions hold, then $\sigma(\Theta) = \mathcal{F} = \tau(\underline{X})$, a.s.
3. If condition (a) holds for both Θ_1 and Θ_2 then these Θ's are *comonotone*: There exists an almost-sure set C on which
$$\Theta_1(v) > \Theta_1(w) \;\Rightarrow\; \Theta_2(v) \geq \Theta_2(w).$$

<u>*What does this mean?*</u>

- This gives a de Finetti-style characterization of the *monotone unidimensional representation.*
- Part (c) [CA and VCD] gives "observable" criteria for including monotone unidimensional representations in a statistical model.

Main ideas of the proof

$(a) \Rightarrow (c)$: Essentially done on the page introducing CA and VCD!

$(c) \Rightarrow (b)$: Take $\mathcal{F} = \tau(\underline{X})$ and use standard approximation arguments.

$(b) \Rightarrow (a)$: *The hard part!* We construct a $\Theta \in \mathbb{R}$ and show that CI and M hold for this Θ...

Central ideas for $(b) \Rightarrow (a)$:

Suppose there is an $\mathcal{F}$ satisfying the conditions of part (b).

- Define a set of *true scores* $\underline{T} = (T_{iq} : i \in \mathbb{N}, q \in \mathcal{Q})$, where $T_{iq} = P[X_i > q|\mathcal{F}]$, and let
$$\tau(\underline{T}) = \cap_{n=1}^{\infty} \sigma(T_{iq} : i > n, q \in \mathcal{Q}).$$

- $\tau(\underline{T})$ provides a bridge between $\mathcal{F}$ and $\tau(\underline{X})$.

- The true scores T_{iq} are all comonotone with one another; and hence can be used to construct an appropriate Θ.

The true scores

$$T_{iq} = P[X_i > q \mid \mathcal{F}]$$

Consider

$$Y_{iq} = \begin{cases} 1, & \text{if } X_i > q; \\ 0, & \text{if } X_i \leq q. \end{cases}$$

so that

$$T_{iq} = P[X_i > q \mid \mathcal{F}] = E[Y_{iq} \mid \mathcal{F}]$$

- Y_{iq} are often used to analyze polytomous response data (the "item steps"). We use them to analyze arbitrary (even continuous) response data.

- If $\mathcal{F}$ is replaced with a latent trait Θ, then

  $$T_{iq} = P[X_i > q \mid \theta] = E[Y_{iq} \mid \theta]$$

  are the "ISRF's" in a nonparametric polytomous analysis.

- *Everything* that can be known about the "latent behavior" of the X_i's (conditional on $\mathcal{F}$) *can be learned from the true scores* T_{iq}.

The Proof

<u>Lemma 1</u>

Suppose $\mathcal{F} \subseteq \tau(\underline{X})$ and $\amalg \underline{X}|\mathcal{F}$. Then

$$\mathcal{F} = \sigma(\underline{T}) = \tau(\underline{T}) = \tau(\underline{X}), a.s.$$

<u>*Interpretation*</u>

- If $\mathcal{F}$ is rich enough to induce CI, then $\mathcal{F} = \tau(\underline{X})$.
- The true scores T_{iq} are quite redundant, in the sense that $\sigma(\underline{T}) = \tau(\underline{T})$.

<u>Lemma 2</u>

Suppose $\mathcal{F} \subseteq \tau(\underline{X})$; $\amalg \underline{X}|\mathcal{F}$; and the covariance condition $(*)$ of part (b) of the theorem holds. Then each pair (T_{iq}, T_{jr}) is comonotone.

<u>*Interpretation*</u>

- Condition $(*)$, related to CA, severely constrains the variation of the T_{iq}'s.
- Since every pair (T_{iq}, T_{jr}) is comonotone, we go hunting for a common variable Θ such that $T_{iq} = f_{iq}(\Theta)$. Our quarry is

$$\Theta = \sum_{i=1}^{\infty} \sum_{q \in \mathcal{Q}} a_{iq} T_{iq}$$

for any summable sequence $a_{iq} > 0$ (and a fixed ordering of $\mathcal{Q}$).

Lemma 1

Suppose $\mathcal{F} \subseteq \tau(\underline{X})$ and $\coprod \underline{X} | \mathcal{F}$. Then

$$\mathcal{F} = \sigma(\underline{T}) = \tau(\underline{T}) = \tau(\underline{X}), a.s.$$

Interpretation

- If $\mathcal{F}$ is rich enough to induce CI, then $\mathcal{F} = \tau(\underline{X})$.

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Proof

For the first "=", "$\supseteq$" follows from the definition of T_{iq}, and "$\subseteq$" is a kind of 0/1 law that can be built up from cylinder sets in $\sigma(\underline{X})$.

The second "=" follows, since the first is still true if you drop a few X's [because $\mathcal{F} \subseteq \tau(\underline{X})$].

For the third "=", "$\subseteq$" is again from the definition of T_{iq}, and "$\supseteq$" is the elementary 0/1 law for independent r.v.'s, given $\mathcal{F}$. $\square$

Lemma 2

Suppose $\mathcal{F} \subseteq \tau(\underline{X})$; $\amalg\, \underline{X}|\mathcal{F}$; and the covariance condition $(*)$ of part (b) of the theorem holds. Then each pair (T_{iq}, T_{jr}) is comonotone.

Interpretation

- Condition $(*)$, related to CA, severely constrains the variation of the T_{iq}'s.

- Since every pair (T_{iq}, T_{jr}) is comonotone, this suggests that we look for a common variable Θ such that $T_{iq} = f_{iq}(\Theta)$.

Proof

$i \neq j$: By conditioning on $\mathcal{F}$ it follows that $\mathrm{Cov}\,(T_{iq}, T_{jr}|A) \geq 0$ for all $A \in \mathcal{F}$ for which the covariance is defined. Consider the almost-sure set $C = \{(s,t) : P[(T_{iq}, T_{jr}) \in B_\delta(s,t)] > 0, \forall \delta > 0\}$: by geometric arguments one can show that C cannot contain (s_1, t_1), (s_2, t_2) with $s_1 < s_2$ and $t_1 > t_2$.

$i = j$: By **Lemma 1**, $\exists\, k > i$ such that $T_{iq}, T_{ir} \in \sigma(T_{js} : j > k, s \in \mathcal{Q})$. Then $T_{iq}(v) > T_{ir}(w) \Rightarrow \exists\, T_{js}(v) \neq T_{js}(w) \Rightarrow T_{js}(v) > T_{js}(w)$ $\Rightarrow T_{ir}(v) \geq T_{ir}(w)$ [by the case $i \neq j$]. $\Box$

Now construct Θ

For any particular ordering of the rationals q, let $a_{iq} > 0$ such that $\Sigma_{i=1}^{\infty} \Sigma_q a_{iq} < \infty$, and define

$$\Theta = \sum_{i=1}^{\infty} \sum_q a_{iq} T_{iq}.$$

Let C be a common almost-sure set on which all pairs (T_{iq}, T_{jr}) are comonotone (available by countable applications of **Lemma 2**).

Loose ends:

(1) On C,

$$\begin{aligned} \Theta(v) > \Theta(w) &\Rightarrow \exists\, T_{iq}(v) > T_{iq}(w) \\ &\Rightarrow \forall j, r,\ T_{iq}(v) \geq T_{iq}(w). \end{aligned}$$

It follows that *each T_{iq} is a monotone function of Θ, a.s.*

(2) By (1), $T_{iq} \in \sigma(\Theta)$ a.s., so from **Lemma 1**,

$$\mathcal{F} = \sigma(\underline{T}) = \tau(\underline{T}) = \tau(\underline{X}) \subseteq \sigma(\Theta) \subseteq \mathcal{F},\ a.s.$$

Hence

(M): $T_{iq} = P[X_i > q|\mathcal{F}] = P[X_i > q|\Theta]$ is nondecreasing in Θ;
(CI): $\coprod \underline{X}|\mathcal{F} \Rightarrow \coprod \underline{X}|\Theta$.

(3) If Θ_1 and Θ_2 are two random variables satisfying part (a) of the theorem, we can use **Lemma 1** with $\mathcal{F} = \sigma(\Theta_i)$, for each $i = 1, 2$, to show that $\sigma(\Theta_1) = \sigma(\Theta_2)$, a.s., and hence the true scores T_{iq} are equal. Now using the monotonicity assumption in (a), all pairs (Θ_1, T_{iq}) and (Θ_2, T_{iq}) must be comonotone; hence

$$\begin{aligned} \Theta_1(v) > \Theta_1(w) &\Rightarrow \exists\, T_{iq}(v) > T_{iq}(w) \\ &\Rightarrow \Theta_2(v) \geq \Theta_2(w). \end{aligned}$$

The Examples

Example: Linear Factor Analysis

d = 1: $X_j = a_j T + \epsilon_j$ $\qquad$ (T, $\epsilon_j \sim$ i.i.d. $N(0,1)$; $a_j \geq 0$)

$$\not\Sigma_{(X_1,\ldots,X_J)\cdot(X_{J+1},\ldots,X_{J+m})} = I_{J\times J} + \frac{1}{1+\sum_{j=J+1}^{J+m} a_j^2} \begin{bmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_J \\ a_2a_1 & a_2^2 & \cdots & a_2a_J \\ \vdots & \vdots & \ddots & \vdots \\ a_Ja_1 & a_Ja_2 & \cdots & a_J^2 \end{bmatrix}.$$

- CA $\Rightarrow$ Cov$\,(X_i, X_j|$any subset of X_k's$) \geq 0$
- Positive partial correlations $\equiv$ MTP_2.
  - MTP_2 $\Rightarrow$ association .... *but not conversely.*
  - For normals, CA $\Rightarrow$ MTP_2 .... *Converse?*
- VCD $\Rightarrow$ Cov$\,(X_i, X_j|X_{J+1}, \ldots, X_{J+m}) \to 0$; $\Rightarrow \Sigma_{j=1}^J a_j^2 \to \infty$. Then T can be estimated by $\Sigma_{j=1}^J a_j X_j / \Sigma_{j=1}^J a_j^2$, which ensures $T \in \tau(\underline{X})$.

d = 2: $X_j = a_j T_1 + b_j T_2 + \epsilon_j$ (T_i, $\epsilon_j \sim$ i.i.d. $N(0,1)$; a_j, $b_j \geq 0$)

- If $a_1 = 1$, $a_2 = 0$, $b_1 = 0$, $b_2 = 1$, and $a_j = b_j = 1$ for all $j > 2$, then Cov$\,(X_1, X_2|X_3 + \ldots + X_J = c) \leq 0$, violating CA.
- While $\exists$ U such that $\coprod \underline{X}|U$ (Suppes and Zanotti, 1981), $\nexists$ Θ such that CI and M hold, *even if we use a non-linear-regression likelihood and a nonnormal* Θ *distribution!*

Example: The Rasch Model

For $x_j = 0$ or 1, and logit $P_j(\theta) = \theta + \beta_j$,

$$P(x_1, \ldots, x_J) = \int \prod_{j=1}^{J} P_j(\theta)^{x_j} [1 - P_j(\theta)]^{1-x_j} \, dF(\theta).$$

• Log-linear representation:

$$\log P(x_1, \ldots, x_J) = \alpha + \sum_{j=1}^{J} \beta_j x_j + \sum_{k=0}^{J} \gamma_k 1_{\{x_+ = k\}},$$

CA $\Rightarrow$ conditions on the γ_k's (worked out, from a different p.o.v., by Cressie and Holland, 1983).

• VCD $\Rightarrow$

$$p(x_1, \ldots, x_J | x_{J+1}, \ldots, x_{J+m}) \approx \prod_{j=1}^{J} p_i(x_i;\ x_{J+1}, \ldots, x_{J+m}),$$

In the Rasch (or any IRT) model this is easy to see with a Bayesian argument:

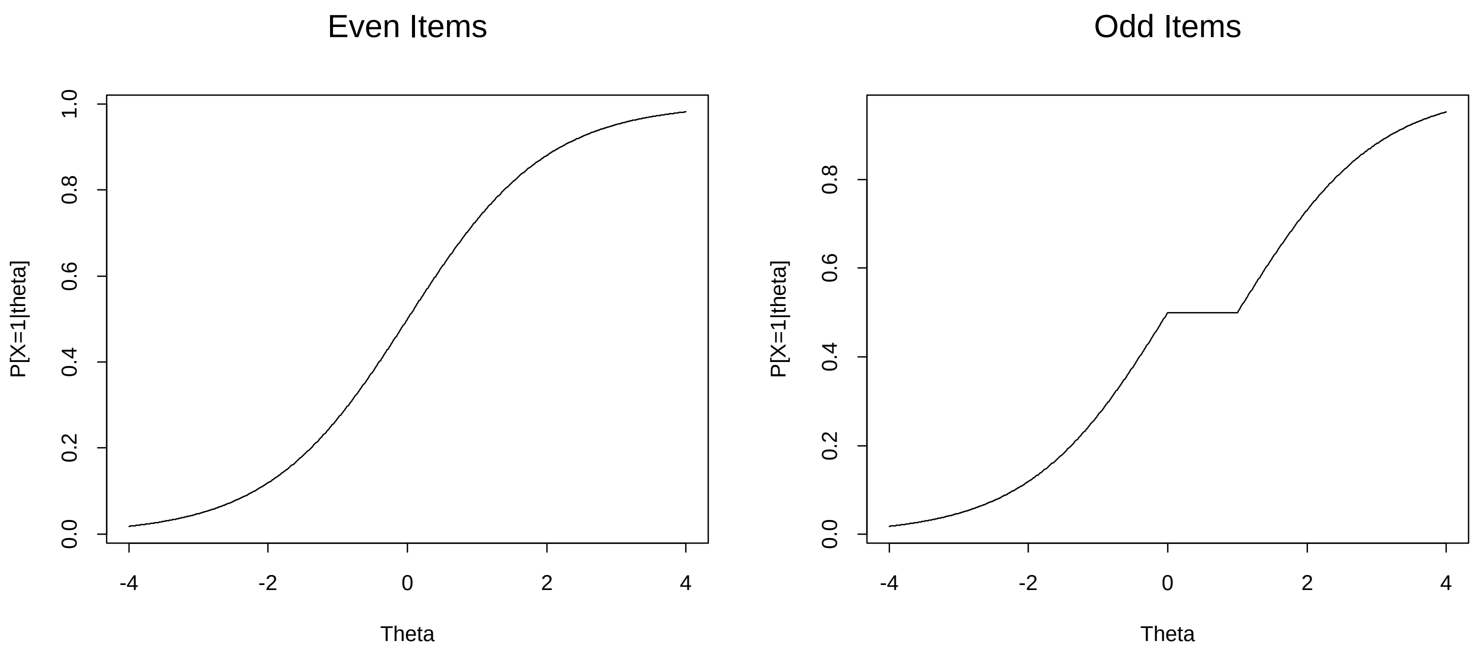
$$p(x_1, \ldots, x_J | x_{J+1}, \ldots, x_{J+m}) = \int \prod_{j=1}^{J} P_j(\theta)^{x_j} [1 - P_j(\theta)]^{1-x_j} \, dF(\theta | x_{J+1}, \ldots, x_{J+m})$$

and $dF(\theta | x_{J+1}, \ldots, x_{J+m}) \longrightarrow 1_{\{\theta \ = \ \text{examinee's } \theta\}}$.

• Chang and Stout (1993 ***Psy***): this works if the β_j's are bounded on average.

An example where uniqueness "fails"

Consider a $d_L = 1$ representation for $(X_1, X_2, \ldots)$ with ICC's $P_j(\theta)$:



- Whole sequence: Unidimensional with trait Θ.
- Even items: Unidimensional with the same Θ.
- Odd items: Unidimensional with trait Θ'; "compresses out" [0,1].
- $\sigma(\Theta') \in \sigma(\Theta) = \tau(X_1, X_2, X_3, \ldots)$, but $\sigma(\Theta) \notin \sigma(\Theta') = \tau(X_1, X_3, X_5, \ldots)$.
- Possible fix: Require $\Theta \in$ "subtail", the set of all inferences, functions, etc., estimatable from *every* infinite subsequence of X_j's.

(A very strong form of "specific objectivity")

Summary and Conclusions

Interpretations

Summary

- "CA and VCD hold, if and only if there is a $d_L = 1$ (monotone, unidimensional, LI) representation for $\underline{X} = (X_1, X_2, \ldots)$, using a Θ equivalent to $\tau(\underline{X})$."
- Under these conditions, Θ is uniquely determined by the items (ordinal uniqueness). But if you change the items you can change Θ.
- Other $d_L = 1$ representations are possible; CA will hold and VCD will fail. In these representations, Θ *will contain more information than* $\tau(\underline{X})$: $\tau(\underline{X}) \subseteq \sigma(\Theta)$.

A Geometric interpretation; dimensionality of tests

- Think of the true scores T_{iq} as "filling out" a manifold in some possibly infinite dimensional space.
- The dimensionality of the manifold is the dimensionality of the test.

  Similar ideas: Ramsay's (1993) TESTDIST program/analysis.
- CA and VCD together hold, iff the manifold is in fact 1-dimensional, i.e., iff there is a θ such that $T_{iq} \equiv T_{iq}(\theta)$.

Generalizations

- CA: Implied by LI, M, $d = 1$.

- VCD says that

  $$p(x_1, \ldots, x_J \mid x_{J+1}, \ldots, x_{J+m}) \approx \prod_1^J p_j(x_j;\ x_{J+1}, \ldots, x_{J+m}).$$

  VCD should work whenever LI, $d = k$ hold, and the k-dimensional $\theta \in \tau(\underline{X})$. *[See Rasch example.]*

- In a LI, M, $d = k$ VCD will still hold. But CA will fail. *What replaces CA when $d > 1$?*

- Difficulty of nonparametric definitions of $d > 1$.

Connections with other work

- *Holland and Rosenbaum (1986, Ann Stat).*
  - Connections with literature on probability inequalities.
  - CA as a very strong necessary condition for $d_L = 1$.
- *Stout (1990, Psy).*
  - Embed $(X_1, \ldots, X_J)$ into an *infinite item sequence* to fully understand latent structure.
  - Ordinal uniqueness.
- *Junker (1993, Ann Stat).*
  - Partial characterization using $d_E = 1$ and CA.
  - Natural to think about $\tau(\underline{X})$, the consistently estimatable quantities.
- *Ellis and van den Wollenberg (1993, Psy).*
  - Complete characterization of a much stronger model (monotone/local homogeneity; see also Meredith and Millsap (1992)).
  - Connections with the notion of co-monotonicity.
- *Ramsay (1993, TESTDIST).*
  - Nonparametric estimation of the "test manifold" using similarity measures between items.
  - Does not depend on fixed-dimensional latent variable representation.

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