Comparing non-nested models in the search for new physics

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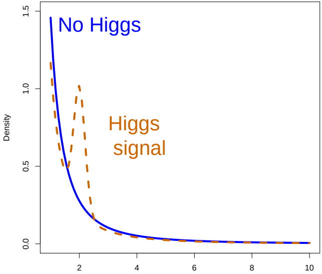
Detection of new physics - The scientific problem

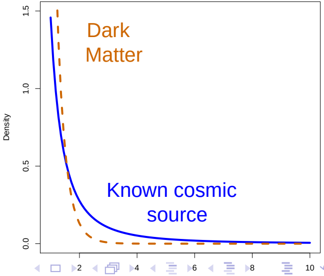


Detection of a new particle

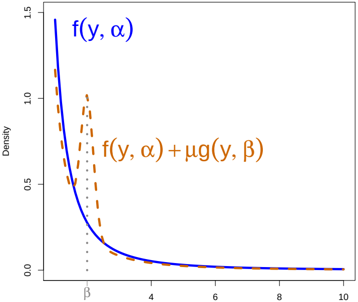
- **E.g.,** Higgs boson, quark, neutrino.
- We want to detect a bump (the signal of the new particle) on top of a background flux.

Distinguish known astrophysics from new signals

- **E.g.,** Dark Matter.
- We can even have a fake signal, i.e., something mimicking Dark Matter, but not a background to it.



Detection of a new particle - The statistical problem



The model of interest is proportional to

$$\underbrace{f(y,\alpha)}_{\text{background}} + \underbrace{\mu}_{\substack{\text{signal}\\\text{strength}}} \underbrace{g(y, \overbrace{\beta}^{\substack{\text{signal}\\\text{location}}})}_{\text{bump}} \qquad (1)$$

and we test

$$H_0 : \mu = 0 \quad \text{vs.} \quad \mu > 0. \qquad (2)$$

Problems

μ is on the boundary of its parameter space + β is not defined under H_0.

Solutions

$\underbrace{\text{Chernoff, 1954 + Davies, 1987}}_{\text{Theoretical solutions}}, \underbrace{\text{Gross and Vitells, 2010}}_{\text{Practical solution}}.$

Testing on the boundary of the parameter space

- **Model:**

$$\propto f(y,\alpha) + \mu g(y,\beta) \qquad \mu \geq 0 \tag{3}$$

For now, let β be fixed, the model in (3) is identifiable.

- **Test**

$$H_0 : \mu = 0 \quad \text{versus} \quad H_1 : \mu > 0$$

- **Test statistics*:**

$$LRT = -2\log[\underbrace{L(0,\hat{\alpha}_0,\text{-})}_{\substack{\text{Likelihood}\\ \text{under } H_0}} - \underbrace{L(\hat{\mu},\hat{\alpha},\beta)}_{\substack{\text{Likelihood}\\ \text{under } H_1}}] \tag{4}$$

* for the specific case of β fixed.

- μ is on the boundary $\Rightarrow$ WE CAN USE Chernoff, 1954 i.e.:

$$LRT \xrightarrow[n\to\infty]{d} \frac{1}{2}\chi^2_1 + \frac{1}{2}\delta(0) \quad \text{under } H_0 \tag{5}$$

Testing with non-identifiable parameters

- If β fixed, under H_0 the LRT is asymptotically $\frac{1}{2}\chi^2_1 + \frac{1}{2}\delta(0)$.
- If we let β vary$\Rightarrow$ Under H_0, $\{LRT(\beta), \beta \in \boldsymbol{B}\}$ is asymptotically a $\frac{1}{2}\chi^2_1 + \frac{1}{2}\delta(0)$ random process indexed by β.
- In practice:
  - Define a grid $\boldsymbol{B}_R$ of R β_r values over the energy spectrum $\boldsymbol{B}$.
  - $\forall \beta_r \in \boldsymbol{B}_R$ calculate $LRT(\beta_r)$.

We combine the R $LRT(\beta_r)$ values in a unique test statistics...

$$c = \max_{\beta_r \in \boldsymbol{B}_R} LRT(\beta_r)$$

... and we produce a global p-value...

$$P(\sup_{\beta \in \boldsymbol{B}} LRT(\beta) > c) \quad (6)$$

...which we must calculate/approximate somehow!

Approximation of $P\left(\sup_{\beta\in\boldsymbol{B}} LRT(\beta) > c\right)$

From **Davies, 1987** we have

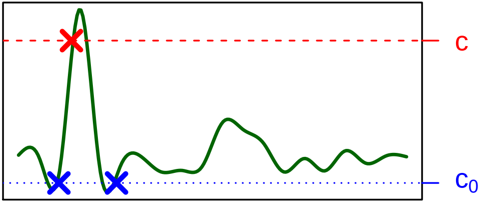
$$P(\sup LRT(\beta) > c) \lessapprox \frac{P(\chi^2_1 > c)}{2} + E[N(c)|H_0] \qquad (7)$$

Expected # of upcrossings over c of the LRT process under H_0

where $\approx$ holds if $c \to +\infty$.
In Davies, 1987

$$E[N(c)|H_0] = \frac{e^{\frac{c}{2}}}{\sqrt{2\pi}} \int_L^U \kappa(\beta) d\beta$$

(not easy to deal with).

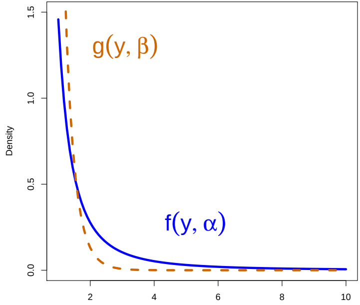


$\Rightarrow$ use the "empirical" version of **Gross and Vitells, 2010**

$$P(\sup LRT(\beta) > c) \lessapprox \frac{P(\chi^2_1 > c)}{2} + e^{-\frac{c-c_0}{2}} E[N(c_0)|H_0] \qquad (8)$$

where $c_0 << c$ and $E[N(c_0)|H_0]$ is estimated using (few) Bootstrap simulations.

Distinguish known astrophysics from new signals - The statistical problem



- The model for the know cosmic source is $f(y,\alpha)$;
- The model for the new source is $g(y,\beta)$;
- $f \not\equiv g$ for any α and β.

Is f sufficient to explain the data, or does g provide a better fit?

Problem

f and g are non-nested.

Solutions

Cox, 1961-1962, Atkinson, 1970; etc. (Theoretical solutions), Bootstrap, **next slides.** (Practical solutions).

Note

In High Energy Physics (HEP) a discovery is claimed at $\mathbf{5}\sigma$ significance. Simulating $O(10^8)$ from a detector might get quite prohibitive.

A new formulation of the problem

Consider a comprehensive model which includes $f(y, \alpha)$ and $g(y, \beta)$ as special cases:

$$(1 - \eta)f(y, \alpha) + \eta g(y, \beta) \tag{9}$$

Thus, considering the model in (9) we test

$$H_0 : \eta = 0 \quad \text{versus} \quad H_1 : \eta > 0$$

To exclude intermediate values of η we can interchange the roles of the hypotheses and test

$$H_0 : \eta = 1 \quad \text{versus} \quad H_1 : \eta < 1.$$

From a new formulation to a well known problem

Model:

$$(1-\underbrace{\eta}_{\substack{\text{Tested}\\\text{on the}\\\text{boundary}}})f(y,\alpha)+\eta g(y,\underbrace{\beta}_{\substack{\text{Not}\\\text{defined}\\\text{under}\\H_0}}) \quad \text{with} \quad 0\leq\eta\leq 1 \qquad (10)$$

Test:

$H_0:\eta=0$ versus $H_1:\eta>0$

similar argument for $H_0:\eta=1$ versus $H_1:\eta<1$

Note!

These are precisely the same issues we encounter when detecting new particles $\Longrightarrow$ **we already have a solution!**

Does it actually work? Let' s see an example...

Null model: Power law (Pareto Type I)

$$f(E,\phi) \propto \phi E^{-(\phi+1)}$$

Alternative model: Dark Matter (from Bergström et al., 1998)

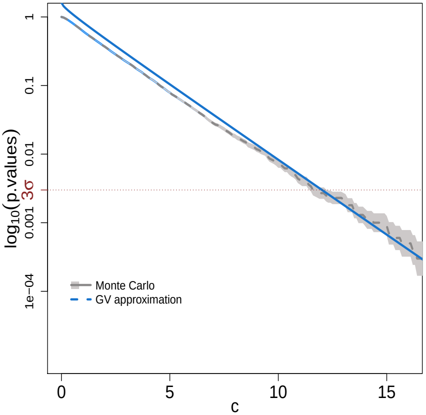
$$g(E,M_\chi) \propto E^{-1.5}\exp\left\{-7.8\frac{E}{M_\chi}\right\}$$

Comprehensive model:

$$(1-\eta)\frac{\phi E^{-(\phi+1)}}{k(\phi)} + \eta\frac{E^{-1.5}}{k(M_\chi)}\exp\left\{-7.8\frac{E}{M_\chi}\right\}$$

where $E, M_\chi \in [1,100]$ and $\phi > 0$.

Test: $H_0 : \eta = 0$ versus $H_1 : \eta > 0$



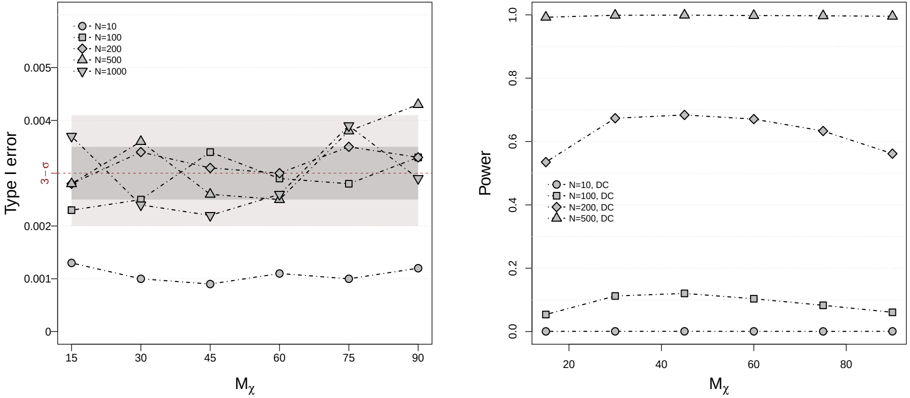
N. of Monte Carlo simulations: 10,000.

Sample size: 10,000.⇐ Large "N"!

More examples in: S. Algeri, J. Conrad and D.A. van Dyk. *A method for comparing non-nested models with application to astrophysical searches for new physics.*. MNRAS: Letters, 2016.

What if we have just few events?

Simulation with “not-that-large” N



For a comparison with other inferential procedure see: Algeri S. et al. *Looking for a Needle in a Haystack? Look Elsewhere!* Submitted, 2016.

Realistic data analysis

- We simulated 200 events a 5 years observation of putative dark matter source from the Fermi Large Area Telescope (LAT).
- The Fermi LAT is a γ-ray telescope on board the earth-orbiting Fermi satellite.

Results[§]

Power-law vs. dark matter

p-value$= 2.7 \cdot 10^{-5}$ (sig. 4.038σ)

Dark matter vs. power-law

p-value$= 0.528$

- To improve the power of the test one could take into account:
  - γ-ray directions.
  - Instrumental error.

Image from: Cowen R. *Space telescope to get software fix.* Nature, Vol. 491, 2012.

§ Using *R package 'NONnest'*, S. Algeri, 2015.

Conclusions and future works

We have presented a two-step solution to a compare competing non-nested models:

- **Step 1** - Extend the parameter space of the models to be compared through an additive comprehensive model.
- **Step 2** - Apply Gross and Vitells, 2010 on the model in Step 1.

Advantages of the procedure

- (Extremely) easy to implement.
- No extensive calculations on a case-by-case basis.
- Computationally more efficient than standard Bootstrap simulations.

Limitations and future works

- It does not handle multi-dimensional nuisance parameter under H_1.
- The nuisance parameter under H_0 is required to lie in the interior of its parameter space.
- Improvement of the GV bound w.r.t the dependence structure of the LRT perocess.

References

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