# Chapter 5

# Stationary One-Parameter Processes

Section 5.1 describes the three main kinds of stationarity: strong, weak, and conditional.

Section 5.2 relates stationary processes to the shift operators introduced in the last chapter, and to measure-preserving transformations more generally.

## 5.1 Kinds of Stationarity

Stationary processes are those which are, in some sense, the same at different times — slightly more formally, which are invariant under translation in time. There are three particularly important forms of stationarity: strong or strict, weak, and conditional.

**Definition 49 (Strong Stationarity)** *A one-parameter process is* strongly stationary *or* strictly stationary *when all its finite-dimensional distributions are invariant under trnaslation of the indices. That is, for all $\tau \in T$, and all $J \in \mathrm{Fin}(T)$,*

$$\mathcal{L}(X_J) = \mathcal{L}(X_{J+\tau}) \quad (5.1)$$

Notice that when the parameter is discrete, we can get away with just checking the distributions of blocks of consecutive indices.

**Definition 50 (Weak Stationarity)** *A one-parameter process is* weakly stationary *or* second-order stationary *when, for all $t \in T$,*

$$\mathbf{E}[X_t] = \mathbf{E}[X_0] \quad (5.2)$$

*and for all $t, \tau \in T$,*

$$\mathbf{E}[X_\tau X_{\tau+t}] = \mathbf{E}[X_0 X_t] \quad (5.3)$$

At this point, you should check that a weakly stationary process has time-invariant correlations. (We will say much more about this later.) You should also check that strong stationarity implies weak stationarity. It will turn out that weak and strong stationarity coincide for Gaussian processes, but not in general.

**Definition 51 (Conditional (Strong) Stationarity)** *A one-parameter process is conditionally stationary if its conditional distributions are invariant under time-translation: $\forall n \in \mathbb{N}$, for every set of $n+1$ indices $t_1, \ldots t_{n+1} \in T$, $t_i < t_{i+1}$, and every shift $\tau$,*

$$\mathcal{L}\left(X_{t_{n+1}} | X_{t_1}, X_{t_2} \ldots X_{t_n}\right) = \mathcal{L}\left(X_{t_{n+1}+\tau} | X_{t_1+\tau}, X_{t_2+\tau} \ldots X_{t_n+\tau}\right) \quad (5.4)$$

*(a.s.).*

Strict stationarity implies conditional stationarity, but the converse is not true, in general. (Homogeneous Markov processes, for instance, are all conditionally stationary, but most are not stationary.) Many methods which are normally presented using strong stationarity can be adapted to processes which are merely conditionally stationary.[1]

Strong stationarity will play an important role in what follows, because it is the natural generaliation of the IID assumption to situations with dependent variables — we allow for dependence, but the probabilistic set-up remains, in a sense, unchanging. This will turn out to be enough to let us learn a great deal about the process from observation, just as in the IID case.

## 5.2 Strictly Stationary Processes and Measure-Preserving Transformations

The shift-operator representation of Section 4.2 is particularly useful for strongly stationary processes.

**Theorem 52 (Stationarity is Shift-Invariance)** *A process $X$ with measure $\mu$ is strongly stationary if and only if $\mu$ is shift-invariant, i.e., $\mu = \mu \circ \Sigma_\tau^{-1}$ for all $\Sigma_\tau$ in the time-evolution semi-group.*

Proof: "If" (invariant distributions imply stationarity): For any finite collection of indices $J$, $\mathcal{L}(X_J) = \mu \circ \pi_J^{-1}$ (Lemma 25), and similarly $\mathcal{L}(X_{J+\tau}) = \mu \circ \pi_{J+\tau}^{-1}$.

$$\pi_{J+\tau} = \pi_J \circ \Sigma_\tau \quad (5.5)$$
$$\pi_{J+\tau}^{-1} = \Sigma_\tau^{-1} \circ \pi_J^{-1} \quad (5.6)$$
$$\mu \circ \pi_{J+\tau}^{-1} = \mu \circ \Sigma_\tau^{-1} \circ \pi_J^{-1} \quad (5.7)$$
$$\mathcal{L}(X_{J+\tau}) = \mu \circ \pi_J^{-1} \quad (5.8)$$
$$= \mathcal{L}(X_J) \quad (5.9)$$

[1] For more on conditional stationarity, see Caires and Ferreira (2005).

"Only if": The statement that $\mu = \mu \circ \Sigma_\tau^{-1}$ really means that, for any set $A \in \mathcal{X}^T$, $\mu(A) = \mu(\Sigma_\tau^{-1} A)$. Suppose $A$ is a finite-dimensional cylinder set. Then the equality holds, because all the finite-dimensional distributions agree (by hypothesis). But this means that $X$ and $\Sigma_\tau X$ are two processes with the same finite-dimensional distributions, and so their infinite-dimensional distributions agree (Theorem 23), and the equality holds on all measurable sets $A$. □

This can be generalized somewhat.

**Definition 53 (Measure-Preserving Transformation)** *A measurable mapping $F$ from a measurable space $\Xi, \mathcal{X}$ into itself preserves measure $\mu$ iff, $\forall A \in \mathcal{X}$, $\mu(A) = \mu(F^{-1}A)$, i.e., iff $\mu = \mu \circ F^{-1}$. This is true just when $F(X) \stackrel{d}{=} X$, when $X$ is a $\Xi$-valued random variable with distribution $\mu$. We will often say that $F$ is measure-preserving, without qualification, when the context makes it clear which measure is meant.*

*Remark on the definition.* It is natural to wonder why we write the defining property as $\mu = \mu \circ F^{-1}$, rather than $\mu = \mu \circ F$. There is actually a subtle difference, and the former is stronger than the latter. To see this, unpack the statements, yielding respectively

$$\forall A \in \mathcal{X},\ \mu(A) = \mu(F^{-1}(A)) \quad (5.10)$$

$$\forall A \in \mathcal{X},\ \mu(A) = \mu(F(A)) \quad (5.11)$$

To see that Eq. 5.10 implies Eq. 5.11, pick any measurable set $B$, and then apply 5.10 to $F(B)$ (which is $\in \mathcal{X}$, because $F$ is measurable). To go the other way, from 5.11 to 5.10, it would have to be the case that, $\forall A \in \mathcal{X}$, $\exists B \in \mathcal{X}$ such that $A = F(B)$, i.e., every measurable set would have to be the image, under $F$, of another measurable set. This is not necessarily the case; it would require, for starters, that $F$ be onto (surjective).

Theorem 52 says that every stationary process can be represented by a measure-preserving transformation, namely the shift. Since measure-preserving transformations arise in many other ways, however, it is useful to know about the processes they generate.

**Corollary 54 (Measure-preservation implies stationarity)** *If $F$ is a measure-preserving transformation on $\Xi$ with invariant measure $\mu$, and $X$ is a $\Xi$-valued random variable, $\mathcal{L}(X) = \mu$, then the sequence $F^n(X)$, $n \in \mathbb{N}$ is strongly stationary.*

PROOF: Consider shifting the sequence $F^n(X)$ by one: the $n^{\text{th}}$ term in the shifted sequence is $F^{n+1}(X) = F^n(F(X))$. But since $\mathcal{L}(F(X)) = \mathcal{L}(X)$, by hypothesis, $\mathcal{L}\left(F^{n+1}(X)\right) = \mathcal{L}(F^n(X))$, and the measure is shift-invariant. So, by Theorem 52, the process $F^n(X)$ is stationary.

## 5.3 Exercises

**Exercise 5.1 (Functions of Stationary Processes)** *Use Corollary 54 to show that if $g$ is any measurable function on $\Xi$, then the sequence $g(F^n(X))$ is also stationary.*

**Exercise 5.2 (Continuous Measure-Preserving Families of Transformations)** *Let $F_t$, $t \in \mathbb{R}^+$, be a semi-group of measure-preserving transformations, with $F_0$ being the identity. Prove the analog of Corollary 54, i.e., that $F_t(X)$, $t \in \mathbb{R}^+$, is a stationary process.*

**Exercise 5.3 (The Logistic Map as a Measure-Preserving Transformation)** *The logistic map with $a = 4$ is a measure-preserving transformation, and the measure it preserves has the density $1/\pi\sqrt{x(1-x)}$ (on the unit interval).*

1. *Verify that this density is invariant under the action of the logistic map.*
2. *Simulate the logistic map with* uniformly *distributed $X_0$. What happens to the density of $X_t$ as $t \to \infty$?*