Chapter 13

The Strong Markov Property and Martingale Problems

Section 13.1 introduces the strong Markov property — independence of the past and future conditional on the state at *random* (optional) times. It includes an example of a Markov process which is not *strongly* Markovian.

Section 13.2 describes "the martingale problem for Markov processes", explains why it would be nice to solve the martingale problem, and how solutions are strong Markov processes.

13.1 The Strong Markov Property

A process is Markovian, with respect to a filtration $\{\mathcal{F}\}_t$, if for any *fixed* time t, the future of the process is independent of $\mathcal{F}_t$ given X_t. This is not necessarily the case for a *random* time τ, because there could be subtle linkages between the random time and the evolution of the process. If these can be ruled out, we have a *strong* Markov process.

Definition 165 (Strongly Markovian at a Random Time) *Let X be a Markov process with respect to a filtration $\{\mathcal{F}\}_t$, with transition kernels $\mu_{t,s}$ and evolution operators $K_{t,s}$. Let τ be an $\{\mathcal{F}\}_t$-optional time which is almost surely finite. Then X is* strongly Markovian at τ *when either of the two following (equivalent) conditions hold*

$$\mathbb{P}\left(X_{t+\tau} \in B|\mathcal{F}_\tau\right) = \mu_{\tau,\tau+t}(X_\tau, B) \tag{13.1}$$
$$\mathbf{E}\left[f(X_{\tau+t})|\mathcal{F}_\tau\right] = (K_{\tau,\tau+t}f)(X_\tau) \tag{13.2}$$

for all $t \geq 0$, $B \in \mathcal{X}$ and bounded measurable functions f.

Definition 166 (Strong Markov Property) *If X is Markovian with respect to $\{\mathcal{F}\}_t$, and strongly Markovian at every $\{\mathcal{F}\}_t$-optional time which is almost surely finite, then it is a* strong Markov process *with respect to $\{\mathcal{F}\}_t$.*

If the index set T is discrete, then the strong Markov property is implied by the ordinary Markov property (Definition 102). If time is continuous, this is not necessarily the case. It is generally true that, if X is Markov and τ takes on only *countably* many values, X is strongly Markov at τ (Exercise 13.1). In continuous time, however, the Markov property does not imply the strong Markov property.

Example 167 (A Markov Process Which Is Not Strongly Markovian) *(After Fristedt and Gray (1997, pp. 626–627).) We will construct an $\mathbb{R}^2$-valued Markov process on $[0,\infty)$ which is not strongly Markovian. Begin by defining the following map from $\mathbb{R}$ to $\mathbb{R}^2$:*

$$f(w) = \begin{cases} (w,0) & w \leq 0 \\ (\sin w, 1-\cos w) & 0 < w < 2\pi \\ (w-2\pi, 0) & w \geq 2\pi \end{cases} \tag{13.3}$$

When w is less than zero or above 2π, $f(w)$ moves along the x axis of the plane; in between, it moves along a circle of radius 1, centered at $(0,1)$, which it enters and leaves at the origin. Notice that f is invertible everywhere except at the origin, which is ambiguous between $w=0$ and $w=2\pi$.

Let $X(t) = f(W(t)+\pi)$, where $W(t)$ is a standard Wiener process. At all t, $\mathbb{P}(W(t)+\pi=0) = \mathbb{P}(W(t)+\pi=2\pi) = 0$, so, with probability 1, $X(t)$ can be inverted to get $W(t)$. Since $W(t)$ is a Markov process, it follows that $\mathbb{P}(X(t+h) \in B|X(t)=x) = \mathbb{P}\left(X(t+h)\in B|\mathcal{F}_t^X\right)$ almost surely, i.e., X is Markov. Now consider $\tau = \inf_t X(t) = (0,0)$, the hitting time of the origin. This is clearly an F^X-optional time, and equally clearly almost surely finite, because, with probability 1, $W(t)$ will leave the interval $(-\pi,\pi)$ within a finite time. But, equally clearly, the future behavior of X will be very different if it hits the origin because $W=\pi$ or because $W=-\pi$, which cannot be determined just *from X. Hence, there is at least one optional time at which X is not strongly Markovian, so X is not a strong Markov process.*

Since we often want to condition on the state of the process at random times, we would like to find conditions under which a process is strongly Markovian for all optional times.

13.2 Martingale Problems

One approach to getting strong Markov processes is through martingales, and more specifically through what is known as the martingale problem.

Notice the following consequence of Theorem 158:

$$K_t f(x) - f(x) = \int_0^t K_s G f(x) ds \tag{13.4}$$

for any $t \geq 0$ and $f \in \mathrm{Dom}(G)$. The relationship between $K_t f$ and the conditional expectation of f suggests the following definition.

Definition 168 (Martingale Problem) *Let Ξ be a Polish space, $\mathcal{D}$ a class of bounded, continuous, real-valued functions on Ξ, and G an operator from $\mathcal{D}$ to bounded, measurable functions on Ξ. A Ξ-valued stochastic process on $\mathbb{R}^+$ is a* solution to the martingale problem for G and $\mathcal{D}$ *if, for all $f \in \mathcal{D}$,*

$$f(X_t) - \int_0^t Gf(X_s)ds \tag{13.5}$$

is a martingale with respect to $\left\{\mathcal{F}^X\right\}_t$, the natural filtration of X.

Proposition 169 (Cadlag Nature of Functions in Martingale Problems) *Suppose X is a cadlag solution to the martingale problem for $G, \mathcal{D}$. Then for any $f \in \mathcal{D}$, the stochastic process given by Eq. 13.5 is also cadlag.*

PROOF: Follows from the assumption that f is continuous. □

Lemma 170 (Alternate Formulation of Martingale Problem) *X is a solution to the martingale problem for $G, \mathcal{D}$ if and only if, for all $t, s \geq 0$,*

$$\mathbf{E}\left[f(X_{t+s})|\mathcal{F}_t^X\right] - \mathbf{E}\left[\int_t^{t+s} Gf(X_u)du|\mathcal{F}_t^X\right] = f(X_t) \tag{13.6}$$

PROOF: Take the definition of a martingale and re-arrange the terms in Eq. 13.5. □

Martingale problems are important because of the two following theorems (which can both be refined considerably).

Theorem 171 (Markov Processes Solve Martingale Problems) *Let X be a homogeneous Markov process with generator G and cadlag sample paths, and let $\mathcal{D}$ be the continuous functions in* $\mathrm{Dom}(G)$. *Then X solves the martingale problem for $G, \mathcal{D}$.*

PROOF: Exercise 13.2. □

Theorem 172 (Solutions to the Martingale Problem are Strongly Markovian) *Suppose that for each $x \in \Xi$, there is a unique cadlag solution to the martingale problem for $G, \mathcal{D}$ such that $X_0 = x$. Then the collection of these solutions is a homogeneous strong Markov family X, and the generator is equal to G on $\mathcal{D}$.*

PROOF: Exercise 13.3. □

The main use of Theorem 171 is that it lets us prove convergence of some functions of Markov processes, by showing that they can be cast into the form of Eq. 13.5, and then applying the martingale convergence devices. The other use is in conjunction with Theorem 172. We will often want to show that a *sequence* of Markov processes converges on a limit which is, itself, a Markov process. One approach is to show that the terms in the sequence solve martingale problems (via Theorem 171), argue that then the limiting process does too, and finally invoke Theorem 172 to argue that the limiting process must itself be strongly Markovian. This is often *much* easier than showing directly that the limiting process is Markovian, much less strongly Markovian. Theorem 172 itself is often a convenient way of showing that the strong Markov property holds.

13.3 Exercises

Exercise 13.1 (Strongly Markov at Discrete Times) *Let X be a homogeneous Markov process with respect to a filtration $\{\mathcal{F}\}_t$ and τ be an $\{\mathcal{F}\}_t$-optional time. Prove that if $\mathbb{P}(\tau < \infty) = 1$, and τ takes on only countably many values, then X is strongly Markovian at τ. (*Note: *the requirement that X be homogeneous can be lifted, but requires some more technical machinery I want to avoid.)*

Exercise 13.2 (Markovian Solutions of the Martingale Problem) *Prove Theorem 171. Hints: Use Lemma 170, bounded convergence, and Theorem 158.*

Exercise 13.3 (Martingale Solutions are Strongly Markovian) *Prove Theorem 172. Hint: use the Optional Sampling Theorem (from 36-752, or from chapter 7 of Kallenberg).*