5.1 Review

We begin by quickly re-defining some concepts from previous lectures. Specify a motif $f$ on $k$ nodes, and a graph $g$ on $n$ nodes.

**Definition 5.1 (Homomorphism Density)** The **homomorphism density** of $f$ into $g$ is

$$t(f, g) = \frac{\text{Hom}(f, g)}{n^k}$$  \hspace{1cm} (5.1)

where $\text{Hom}(f, g)$ counts the number of homomorphisms from $f$ into $g$.

Recall that for any graph $g$ with corresponding adjacency matrix $a$, we can define a function $w_g : [0, 1] \times [0, 1] \rightarrow \{0, 1\}$ as follows:

$$w_g(u, v) = a_{[nu][nv]}$$  \hspace{1cm} (5.2)

We can think of $w_g$ as taking the adjacency matrix and squashing it into the unit square, and sampling a graph from $w_g$ is the same as picking nodes from $g$, then connecting them if there is an edge between them in $g$. This leads to a second (and equivalent) definition for $t(f, g)$,

$$t(f, g) = \int_{[0,1]^k} \prod_{(i,j) \in E(f)} w_g(u_i, u_j) du_1 \ldots du_k$$  \hspace{1cm} (5.3)

Recall the injective homomorphism density $t_{inj}(f, g) = \mathbb{P}(f \preceq G[k])$ where $G[k]$ is the induced subgraph we get by randomly sampling $k$ nodes from $g$. The following proposition bounds the difference between $t$ and $t_{inj}$.

**Proposition 5.2**

$$|t(f, g) - t_{inj}(f, g)| \leq \frac{k^2}{2n}$$  \hspace{1cm} (5.4)
Notice that we can define the homomorphism density for a function \( w : [0, 1] \times [0, 1] \to [0, 1] \) in a similar manner as we did for a graph.

\[
t(f, w) = \int_{[0,1]^k} \prod_{(i,j) \in E(f)} w(u_i, u_j) du_1 ... du_k
\]  

(5.5)

and so \( t(f, g) = t(f, w_g) \). This allows us to talk about the convergence of graph sequences to a graphon function.

**Definition 5.3 (Convergence of Graph Sequences)** A sequence \( g_1, g_2, ..., g_m, ... \) converges when \( \forall f, t(f, g_m) \to t(f, w) \). The sequence converges to \( w \) when \( \forall f, t(f, g_m) \to t(f, w) \) (5.6)

Now we are ready to define a graphon.

**Definition 5.4 (Graphon)** Two functions \( w_1 \) and \( w_2 \) are equivalent iff \( \forall f, t(f, w_1) = t(f, w_2) \). An equivalence class of \( w \)s is called a graphon.

For any graphon function, the \( w \)-random graph on \( n \) nodes, \( G_n(w) \), is the Conditionally Independent Dyad model with \( w \) as the edge probability function.

Finally, we come to the main question for today: does \( G_n(w) \to w \) in any useful sense? Recall that we might be interested in any of the four main types of convergence for random quantities: convergence in probability, almost sure convergence, convergence in distribution, and convergence in squared mean. We will focus on the first two today.

### 5.2 Convergence of \( w \)-random graphs

Our first theorem will cover convergence in probability of \( G_n(w) \) to \( w \).

**Theorem 5.5** For any motif \( f \) and any \( \epsilon \in (0, 1) \),

\[
Pr(|t(f, G_n(w)) - t(f, w)| > \epsilon) \leq 2e^{-\frac{\epsilon^2 n}{k^2}}
\]

(5.7)

**Proof:** To begin with, recall that one way to generate \( G_n(w) \) is to say that \( (i, j) \) appears in \( G_n(w) \) if \( \xi_{ij} > w(U_i, U_j) \) where the \( \xi_{ij} \) and \( U_i \) are independent draws from \( U(0, 1) \). Let \( Z_i = \{U_i, \xi_{1,i}, ..., \xi_{i-1,i}\} \). Then, changing \( Z_i \) changes the value of \( t(f, G_n(w)) \) by at most \( \frac{k}{n} \). We can therefore use the Bounded Difference Inequality and say

\[
P \left( |t(f, G_n(w)) - E[t(f, G_n(w))]| > \epsilon \right) \leq 2 \exp \left\{ -\frac{\epsilon^2 n}{k^2} \right\}
\]

(5.8)
We then need to bound \( E[t(f,G_n(w))] - t(f,w) \). To begin with, we use the previously stated fact that for any graph \( g \) on \( n \) nodes,

\[
|t(f,g) - t_{inj}(f,g)| \leq \frac{k^2}{2n} \tag{5.9}
\]

Also,

\[
E[t_{inj}(f,G_n(w))] = t(f,w) \tag{5.10}
\]

Putting these together yields

\[
|E[t(f,G_n(w))] - t(f,w)| \leq \frac{k^2}{2n} \tag{5.11}
\]

Moreover, we have that if \( 2 \exp\{-\frac{\epsilon^2 n}{4k^2}\} \leq 1 \) and \( \epsilon \in (0,1) \), then

\[
\frac{k^2}{2n} \leq \frac{\epsilon^2}{4\log 2} \leq \frac{\epsilon}{2} \tag{5.12}
\]

Thus, we have that \( |E[t(f,G_n(w))] - t(f,w)| \leq \frac{\epsilon}{2} \), and by extension,

\[
P(|t(f,G_n(w)) - t(f,w)| > \epsilon) \leq P\left(|t(f,G_n(w)) - E[t(f,G_n(w))]| > \frac{\epsilon}{2}\right) \tag{5.13}
\]

\[
\leq 2 \exp\{-\frac{\epsilon^2 n}{4k^2}\} \tag{5.14}
\]

A simple consequence of this theorem is that \( \forall f, t(f,G_n(w)) \xrightarrow{p} t(f,w) \). However, we can show something even stronger.

**Corollary 5.6** For each \( f, t(f,G_n(w)) \xrightarrow{a.s} t(f,w) \).

**Proof:** Since the previous deviation inequality decreases exponentially in \( n \), we have that \( \forall \epsilon > 0 \)

\[
\sum_{n=1}^{\infty} P(|t(f,G_n(w)) - t(f,w)| > \epsilon) < \infty \tag{5.15}
\]

so by the Borel-Cantelli Lemma, we have almost sure convergence. In particular, the Borel-Cantelli Lemma gives us
\[ P \left( \cap_{m=0}^{\infty} \left\{ |t(f,G_n(w)) - t(f,w)| > 2^{-m}\text{f.o.} \right\} \right) = 1 \]  
\[ (5.16) \]

which is equivalent to \( P \left( t(f,G_n(w)) \to t(f,w) \right) = 1. \)  

This leads us to a strong Law of Large Numbers for graphs.

**Theorem 5.7 (LLN for graphs)** \( G_n(w) \overset{a.s.}{\to} w \)

**Proof:** We want to show that \( P (\forall f, t(f,G_n(w)) \to t(f,w)) = 1. \) Let \( B_k = \{ f : |V(f)| = k, t(f,G_n(w)) \not\to t(f,w) \} \). Then, \( \forall k > 0, P(B_k) = 0, \) and there are countably many \( B_k, \) so \( P(\cup_{k=1}^{\infty} B_k) = 0. \)  

\[ \square \]