Statistical Computing (36-350)

Lecture 16: Simulation I: Generating Random Variables

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Agenda

- The basic random-variable commands
- Transforming uniform random variables to other distributions
  - The quantile method
  - The rejection method
- Where the uniform random numbers come from

REQUIRED READING: Textbook, chapter 5
OPTIONAL READING: *R Cookbook*, sections 8.2–8.7
Chambers, section 6.10

Stochastic Simulation

Why simulate?

- We want to see what a probability model actually does
- We want to understand how our procedure works on a test case
- We want to use a partly-random procedure

All of these require drawing random variables from distributions

Built-in Random Variable Generators

`runif`, `rnorm`, `rbinom`, `rpois`, `rexp`, etc. etc.
First argument is always `n`, number of variables to generate
Subsequent arguments are parameters to distribution
Parameters are recycled:

```
> rnorm(n=4,mean=c(-1000,1000),sd=1)
[1]  -999.3637  1000.4710 -1000.4449  1000.1040
```

sample

```
sample(x, size, replace=FALSE, prob=NULL)
```

draw random sample of `size` points from `x`, optionally with replacement and/or weights
`x` can be anything where `length()` makes sense, basically
`sample(x)` does a random permutation

```
sample(cats$Sex) # Randomly shuffle sexes among cats
```

If `x` is a single number, treat it like `1:x`

```
> sample(5)
[1] 1 4 3 2 5
```

The Quantile Transform Method

Given: uniform random variable U, CDF F
Claim: $X = F^{-1}(U)$ is a random variable with CDF F
Proof:

$$\mathbb{P}(X \leq a) = \mathbb{P}(F^{-1}(U) \leq a) = \mathbb{P}(U \leq F(a)) = F(a)$$

F^{-1} is the quantile function
So if we can generate uniforms and we can calculate quantiles, we can generate non-uniforms

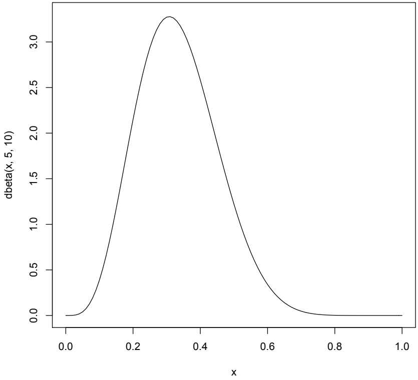
Less Mathematically

To turn U into a coin-toss with bias p: is $U \leq p$ or not?
To turn U into a binomial: start with $X = 0$; if $U \leq F(X)$, stop, otherwise add 1 to X and check again
Tedious do this iteratively
No next value for continuous random variables
Quantiles solve both difficulties

Quantile functions often don't have closed form, and don't have nice numerical solutions

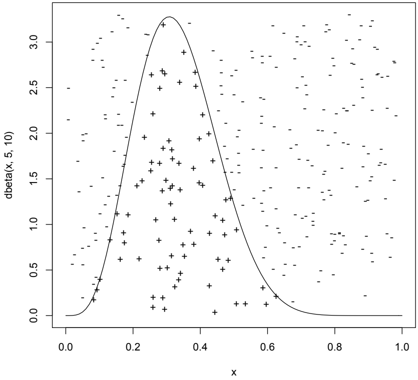
But we know the probability density function — can we use that?

Suppose the pdf f is zero outside an interval $[c,d]$, and $\leq M$ on the interval
Draw the rectangle $[c,d] \times [0,M]$, and the curve f
Area under the curve $= 1$
Area under curve and $x \leq a$ is $F(a)$
How can we uniformly sample area under the curve?

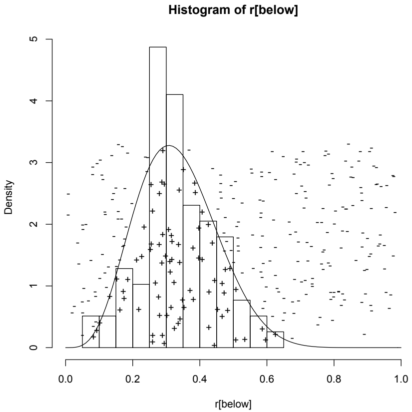



```
M <- 3.3; curve(dbeta(x,5,10),from=0,to=1,ylim=c(0,M))
```

We sample uniformly from the *box*, and take the points under the curve
$R \sim \text{Unif}(c, d)$
$U \sim \text{Unif}(0, 1)$
If $MU \leq f(R)$ then $X = R$, otherwise try again




```
r <- runif(300,min=0,max=1); u <- runif(300,min=0,max=1)
below <- which(M*u <= dbeta(r,5,10))
points(r[below],M*u[below],pch="+"); points(r[-below],M*u[-below],pch="-")
```




```
hist(r[below],xlim=c(0,1),probability=TRUE); curve(dbeta(x,5,10),add=TRUE)
points(r[below],M*u[below],pch="+"); points(r[-below],M*u[-below],pch="-")
```

If f doesn't go to zero outside $[c,d]$, try to find another density ρ where

- ρ also has unlimited support
- $f(a) \leq M\rho(a)$ everywhere
- we can generate from ρ (say by quantiles)

Then $R \sim \rho$, and accept when $MU\rho(R) \leq f(R)$
(Uniformly distributed on the area under ρ)

Need to make multiple "proposals" R for each X
e.g., generated 300 for figure, only accepted 78
Important for efficiency to keep this ratio small
Best way is make sure the proposal distribution is close to the target

Where Do the Uniforms come From?

Uniform numbers are generated by finite algorithms, so really only pseudo-random

We want:

- Number of U_i in $[a, b] \subseteq [0, 1]$ is $\propto (b - a)$
- No correlation between successive U_i
- No detectable dependences in larger or longer groupings

There are now ways of doing a very good job of all of these

Too involved to go into here

Rotations

Take

$$U_{i+1} = U_i + \alpha \mod 1$$

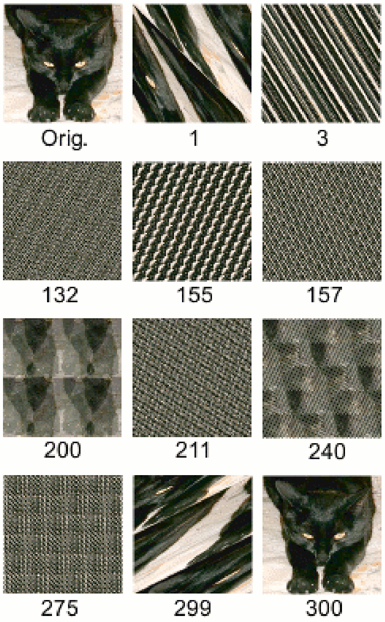
If α is irrational, this never repeats and is uniformly distributed
If α is rational but the denominator is very large, the period is very long, and it is uniform on those points

More Complicated Dynamics

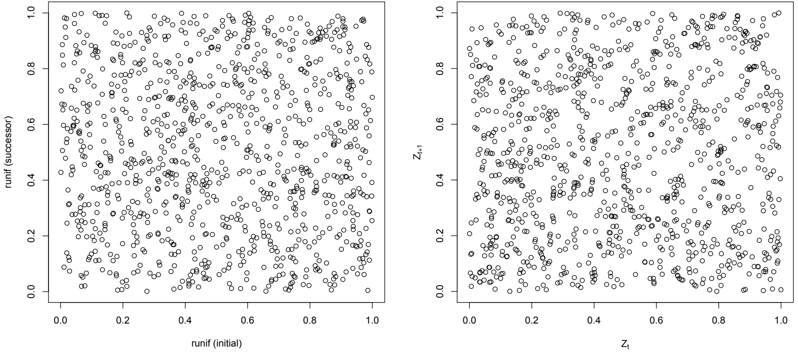
Arnold Cat Map:

$$U_{t+1} = U_t + \phi_t \bmod 1 \tag{1}$$
$$\phi_{t+1} = U_t + 2\phi_t \bmod 1 \tag{2}$$

If we report only U_t, the result is uniformly distributed and hard to predict



Wikipedia, s.v. "Arnold's cat map"



successive values from `runif` vs. Arnold cat map

Nobody uses the cat map, but similar ideas are built into the random-number generator in R (with more dimensions)

Generally: Long periods, rapid divergence of near-by initial conditions (unstable), uniform distribution, low correlation

Using the default generator is a very good idea, unless you really know what you are doing

Setting the Seed

The sequence of pseudo-random numbers depends on the initial condition, or **seed**

Stored in `.Random.seed`, a global variable

To reproduce results exactly, set the seed

```
> old.seed <- .Random.seed # Store the seed
> set.seed(20010805)  # Set it to the day I adopted my cat
> runif(2)
[1] 0.1378908 0.7739319
> set.seed(20010805)  # Reset it
> runif(2)
[1] 0.1378908 0.7739319
> .Random.seed <- old.seed  # Restore old seed
```

See Chambers, §6.10, for some subtleties about working with external programs

Summary

- Unstable dynamical systems give us something very like uniform random numbers
- We can transform these into other distributions when we can compute the distribution function
  - The quantile method when we can invert the CDF
  - The rejection method if all we have is the pdf
- The basic R commands encapsulate a lot of this for us