Statistical Computing (36-350)

Lecture 15: Simulation II: Markov Chains

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Agenda

- Chaining together random variables
- Markov chains
- The long run of Markov chains

READING: Handouts on the class webpage

Multiple Random Variables

`rnorm`, `runif`, etc., give independent and identically distributed (IID) random variables

Most stochastic models don't call for IID random variables

Varying distributions, dependence

How do we generate such things?

Putting the Variables in Order

Try to arrange the variables in order of priority and/or time

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You'll see more of this with graphical models in 36-402

Time Series

Can have a sequence of variables going on in time, $X_1, X_2, \ldots X_n$
Earlier ones can cause later but not other way

$$p(X_1, X_2, \ldots X_n) = p(X_1)p(X_2|X_1)p(X_3|X_2, X_1)\ldots p(X_n|X_{n-1}, X_{n-2}, \ldots X_1)$$

Markov Processes

The **Markov property:** Given the current **state** X_t, earlier states $X_{t-1}, X_{t-2}, \ldots$ are irrelevant to the future states $X_{t+1}, X_{t+2}, \ldots$

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To simulate a Markov chain, we need to

- Draw the initial state X_1 from $p(X_1)$
- Draw X_t from $p(X_t|X_{t-1})$ — inherently sequential

Inputs: number of steps, drawing function for initial distribution, drawing function for transition distribution

```
rmarkov <- function(n,rinitial,rtransition) {
  x <- vector(length=n)
  x[1] <- rinitial()
  for (t in 2:n) {
    x[t] <- rtransition(x[t-1])
  }
  return(x)
}
```

Markov Chains

Each X_t is discrete, not continuous
Represent $p(X_t|X_{t-1})$ in a **transition matrix**,
$\mathbf{q}_{ij} = \Pr(X_t = j | X_{t-1} = i)$
Each row sums to 1 (**stochastic matrix**)

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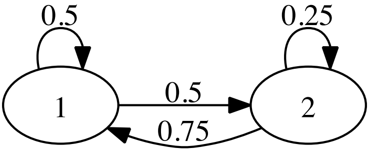
Each row sums to 1 (**stochastic matrix**)
Represent $p(X_1)$ as a vector p_0, summing to 1

Graph vs. matrix



$$\Leftrightarrow q = \left[\begin{array}{cc} 0.5 & 0.5 \\ 0.75 & 0.25 \end{array} \right]$$

Your Basic Markov Chain

```
rmarkovchain <- function(n,p0,q) {
  k <- length(p0)
  stopifnot(k==nrow(q),k==ncol(q),all.equal(rowSums(q),rep(1,time=k)))
  rinitial <- function() { sample(1:k,size=1,prob=p0) }
  rtransition <- function(x) { sample(1:k,size=1,prob=q[x,]) }
  return(rmarkov(n,rinitial,rtransition))
}
```

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  return(rmarkov(n,rinitial,rtransition))
}
```

It runs:

```
> x <- rmarkovchain(1e4,c(0.5,0.5),q)
> head(x)
[1] 1 1 2 1 2 2
```

How do we know it works?

```
> ones <- which(x[-1e4]==1)
> twos <- which(x[-1e4]==2)
> signif(table(x[ones+1])/length(ones),3)
    1     2
0.489 0.511
> signif(table(x[twos+1])/length(twos),3)
    1     2
0.752 0.248
```

vs. $(0.5, 0.5)$ and $(0.75, 0.25)$ ideally
Uses law of large numbers + conditional independence

Hidden Markov Model (HMM)

X_t is Markov, but we see $Y_t = h(X_t) + \text{noise}$, not Markov
e.g.

```
> means <- c(10,-10)
> sds <- c(1,5)
> y <- rnorm(n=length(x),mean=means[x],sd=sds[x])
> signif(head(y),3)
[1]  11.00  10.00 -10.60  11.80 -16.30  -2.41
```

(noise and distortion might be much more complicated)

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Interacting/coupled Markov chains: transition probability for chain 1 depends on its state and chain 2's state

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Interacting/coupled Markov chains: transition probability for chain 1 depends on its state and chain 2's state
Continuous-time Markov chain: stay in the state for a random time, with exponential distribution, then take a chain step
Semi-Markov chain: like CTMC, but non-exponential holding times
Chain with complete connections: as in HMM, $Y_t = h(X_t) + \text{noise}$, but then $X_{t+1} = r(X_t, Y_t)$ (with no noise)

Invariant Distributions

$$\begin{aligned} p_1 &= p_0\mathbf{q} \\ p_2 &= p_1\mathbf{q} = p_0\mathbf{q}^2 \\ p_t &= p_{t-1}\mathbf{q} = p_0\mathbf{q}^t \end{aligned}$$

Fact: If the chain can go from any state to any other and back, and there are no fixed periods, then

$$p_t \rightarrow p_\infty = p_\infty \mathbf{q}$$

$p_\infty =$ left eigenvector of $\mathbf{q}$ of eigenvalue 1
This is the **invariant distribution**

```
> table(rmarkovchain(1e4,c(0.5,0.5),q))
   1    2
5999 4001
> table(rmarkovchain(1e4,c(0.5,0.5),q))
   1    2
5996 4004
> table(rmarkovchain(1e4,c(0,1),q))
   1    2
5989 4011
> table(rmarkovchain(1e4,c(1,0),q))
   1    2
5996 4004
```

```
> eigen(t(q))
$values
[1]  1.00 -0.25

$vectors
          [,1]       [,2]
[1,] 0.8320503 -0.7071068
[2,] 0.5547002  0.7071068

> eigen(t(q))$vectors[,1]/sum(eigen(t(q))$vectors[,1])
[1] 0.6 0.4
```

The Long Run of a Markov Chain

In the long run, all the X_t come close to having the same distribution, the invariant distribution
They're still dependent, though
Ergodic theorem:

$$\frac{1}{n}\sum_{t=1}^{n} f(X_t) \rightarrow \sum_{x} p_{\infty}(x) f(x) = \mathbb{E}_{p_{\infty}}[f(X)]$$

time averages converge on expected values

Summary

1. Break complicated simulations into many draws from basic distributions
    - Make later draws depend on earlier ones
    - Use the Markov property when you can
2. Markov chains are the most basic non-trivial stochastic process
3. In the long run, Markov chains converge on their invariant distribution