Some Problems in Interval Estimation

E. C. Fieller


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Some Problems in Interval Estimation

By E. C. Fieller

SUMMARY

The object of this paper is to propose for discussion the following topic:

\( b_1, b_2, \ldots \) are unbiased estimates of \( \beta_1, \beta_2, \ldots \), distributed normally with variances and covariances jointly estimated, with \( f \) degrees of freedom and independently of \( b_1, b_2, \ldots \), as \( v_{11}, v_{12}, v_{22}, \ldots \), and the functions \( F_i(x) \) do not involve the parameters \( \beta_r \). What can we say about the roots of the equation in

\[
F(\beta, x) = \beta_1 F_1(x) + \beta_2 F_2(x) + \ldots = 0
\]

Numerical examples are discussed in detail to illustrate the problems of determining the fiducial distributions of

(i) the root of a simple equation (i.e., a ratio),
(ii) the roots of a quadratic equation

with variable coefficients. The solutions proposed are based on a consideration of the region of the \((x, t^2)\) plane lying above the curve

\[
(F(b, x))^2 = t^2 V(F(b, x)) = t^2 \Sigma v_{ij} F_i(x) F_j(x).
\]

Introduction

1. My primary purpose this evening is to offer for discussion the following topic:

\( b_1, b_2, \ldots \) are unbiased estimates of \( \beta_1, \beta_2, \ldots \), distributed normally with variances and covariances jointly estimated, with \( f \) degrees of freedom and independently of \( b_1, b_2, \ldots \), as \( v_{11}, v_{12}, v_{22}, \ldots \), and the functions \( F_i(x) \) do not involve the parameters \( \beta_r \). What can we say about the roots of the equation in \( x \)

\[
F(\beta, x) = \beta_1 F_1(x) + \beta_2 F_2(x) + \ldots = 0\tag{1}
\]

I hope to indicate my own views on how this general question should be answered, by considering in detail only the two simplest cases, those in which \( F(\beta, x) \) is (i) of the first degree, (ii) of the second degree in \( x \); but I do not wish to imply that these are the only situations that may be of practical interest.

The Fiducial Distribution of a Ratio

2. As far as I know, Dr. C. I. Bliss (1935a, 1935b) was the first to give, in a particular context, an exact solution in the case in which \( F(\beta, x) \) is of the first degree in \( x \), so that (1) reduces to the form

\[
\eta - \xi x = 0, \tag{2}
\]

and the data to unbiased estimates \( x \) and \( y \) of \( \xi \) and \( \eta \), together with joint estimates \( v_{xx}, v_{xy}, v_{yy} \), independent of \( x \) and \( y \), of their variances and covariance. Discussing the accuracy with which
the probit-log (dose) line could be determined from experimental data, Bliss remarked (1935a, p. 163) that at any chosen probability level the ordinate of the line at abscissa \( X \) could be regarded as lying between the two branches of the Working-Hotelling fiducial hyperbola given, in an obvious notation, by the equation
\[
Y = \hat{y} + b(X - \bar{x}) \pm t(V(\hat{y}) + (X - \bar{x})^2 V(b))^\frac{1}{2},
\]
and inferred (1935b, p. 325) that at the same probability level the abscissa of the line at a given ordinate \( Y \) must also lie within the same area, i.e. in the interval defined by
\[
((Y - \hat{y}) - b(X - \bar{x}))^2 < t^2\{V(Y - \hat{y}) + (X - \bar{x})^2 V(b)\}. 
\]

3. What seems surprising, in retrospect, is that Bliss's work was not immediately recognized—even by Bliss himself—as supplying the answer to the wider question. There is nothing special about the two coefficients \( Y - \hat{y} \) and \( b \) in the first member of (4), when they are looked at as normal deviates, beyond the fact that they are independent; it must therefore follow that (4) supplies the fiducial range for the ratio \( (X - \bar{x}) \) of the expectations of any two independent normal variates \( Y - \hat{y} \) and \( b \), provided that we have suitable estimates of their variances. This is the result that is needed if we wish to find fiducial limits in the usual sorts of biological assay, in which an unknown is tested against a standard preparation. To proceed to the general problem posed by (2), we might note that the covariance of \( x \) and \( (Y - xv_{xy}/v_{xx}) \) is estimated as zero, and that the ratio of their expectations falls short of \( \eta/\xi \) by \( v_{xy}/v_{xx} \); accordingly, (4) gives the fiducial range for \( \eta/\xi \) as the interval defined by
\[
((Y - xv_{xy}/v_{xx}) - x(\alpha - v_{xy}/v_{xx}))^2 < t^2\{V(Y - xv_{xy}/v_{xx}) + (\alpha - v_{xy}/v_{xx})^2 V_{xx}\},
\]
that is, by
\[
(y - \alpha x)^2 < t^2(y_{yy} - 2\alpha v_{xy} + \alpha^2 v_{xx})
\]
or
\[
y^2 - t^2v_{yy} - 2\alpha(xy - t^2v_{xy}) + \alpha^2(x^2 - t^2v_{xx}) < 0.
\]

This same result follows at once, of course, from the remark (Fieller, 1940) that the pivotal quantity
\[
(y - \alpha x)/(v_{yy} - 2\alpha v_{xy} + \alpha^2 v_{xx})\frac{1}{t}
\]
involves only the unknown parameter \( \alpha = \eta/\xi \) and known sufficient statistics \( x, y, v_{xx}, v_{xy}, v_{yy} \), and is distributed as \( t \). Although this pivotal quantity is to be referred to the \( t \) distribution, it is not in general, once \( x, y, v_{xx}, v_{xy}, \) and \( v_{yy} \) are known, a monotonic function of \( \alpha \) capable of assuming all values between \( -\infty \) and \( +\infty \). It is this circumstance, I think, that gives to the problem under discussion its distinctive features.

4. To examine the implications of (5), we may conveniently take \( \alpha \) as abscissa and \( t^2 \) as ordinate, and consider the curve represented by the equation
\[
(y^2 - t^2v_{yy}) - 2\alpha(xy - t^2v_{xy}) + \alpha^2(x^2 - t^2v_{xx}) = 0 
\]
or
\[
t^2 = (y - \alpha x)/(v_{yy} - 2\alpha v_{xy} + \alpha^2 v_{xx})
\]
and
\[
\frac{\alpha^2 v_{xx}}{v_{xx}v_{yy} - v^2_{xy}} - \frac{(y_{yy} - x_{xy}) - (y_{xx} - xv_{xy}) x}{v_{xx}v_{yy} - v^2_{xy}} \frac{(y_{xy} - x_{xy}) v_{xx}}{v_{xx}v_{yy} - v^2_{xy}} \frac{(y_{yy} - 2\alpha v_{xy} + \alpha^2 v_{xx})}{\cdot}
\]
The region of the \( (\alpha, t^2) \) plane lying below this curve may be regarded as barred by (5); for any assigned value of \( t^2 \), the values of \( \alpha \) accepted by (5) are given by that portion of the corresponding parallel to the \( \alpha \)-axis which lies on or above the curve (6). The general nature of this curve is illustrated in Fig. 1, which refers to the familiar Cushny-Peebles data. The curve is a cubic, intersected in two points, real, coincident, or imaginary, by any parallel to the \( \alpha \)-axis; there is one asymptote, the horizontal line \( t^2 = \alpha^2/v_{xx} \) corresponding to the probability-level at which the denominator of the observed ratio \( a = y/x \) ceases to be adjudged significant. The maximum ordinate is, by (8),

\[ t_{\text{max}}^2 = \frac{y^2 v_{xx} - 2 x y v_{xy} + x^2 v_{yy}}{v_{xx} v_{yy}^{-1} - v_{xy}^2} = \frac{x^2}{v_{xx}} + \frac{(y v_{xy} - x v_{yy})^2}{v_{yy}(v_{xx} v_{yy}^{-1} - v_{xy}^2)}, \]  
and is attained at

\[ \alpha = (y v_{xy} - x v_{yy})(y v_{xy} - x v_{yy}^{-1}) = A \text{ say.} \]

For \( t^2 = 0 \) or \( t_{\text{max}}^2 \), the two corresponding values of \( \alpha \) coincide; for \( t^2 > t_{\text{max}}^2 \) they are imaginary, and for \( 0 < t^2 < t_{\text{max}}^2 \) they are given by

\[ \alpha_1, \alpha_2 = \frac{(x y - t^2 v_{xy}) \pm \sqrt{(x y - t^2 v_{xy})^2 - (x^2 - t^2 v_{xx})(y^2 - t^2 v_{yy})}}{x^2 - t^2 v_{xx}}, \]  
the upper sign referring to \( \alpha_1 \), the lower to \( \alpha_2 \). As \( t^2 \) increases from zero through \( x^2/v_{xx} \) to \( t_{\text{max}}^2 \), \( \alpha_2 \) increases (or decreases) steadily from \( a \) through \( (y^2 v_{xx} - x^2 v_{yy})/2x(y v_{xy} - x v_{yy}) \) to \( A \), while \( \alpha_1 \) decreases (or increases) steadily from \( a \), and becomes infinite at \( t^2 = x^2/v_{xx} \) with the sign of \( x(y v_{xy} - x v_{yy}) \); as \( t^2 \) increases through \( x^2/v_{xx} \), \( \alpha_1 \) changes sign at infinity and then steadily approaches \( A \) to rejoin \( \alpha_2 \).

5. The curve (6) is thus in general similar either to that shown in Fig. 1, or else to its mirror image in \( \alpha = 0 \). This fact, I think, renders perfectly natural two features of the problem that have puzzled some workers. The first of them was pointed out to me in 1939 by Dr. J. O. Irwin, when we were corresponding about fiducial limits in biological assay (cf. Irwin, 1943, and for the general case Fieller, 1944). If we regard the fiducial limits for \( \alpha \), when they are real, as determined by (10), they will be inclusive or exclusive according as \( x^2 \geq t^2 v_{xx} \). Secondly, the fiducial range includes infinitely large values of \( \alpha \) whenever \( t^2 v_{xx} \) and if \( t^2 \) is large enough, \( \alpha \) values are excluded. The circumstance that when the denominator of our estimate \( a \) is not significant, the fiducial range for \( \alpha \) includes indefinitely large values is, I think, in accord with what we should intuitively expect; just as is the circumstance that, since the cubic curve (6) cuts the \( t^2 \)-axis at \( t^2 = y^2/v_{yy} \), the fiducial range for \( \alpha \) includes zero when the numerator of \( a \) is not significant. The conclusion that when \( t^2 > t_{\text{max}}^2 \) the fiducial range includes the whole continuum is merely equivalent to the statement that at the corresponding significance level, the data are consistent with all possible hypotheses concerning the value of \( \alpha \). Alternatively, suppose that \( P_{\text{max}} \) is the relative frequency with which, in the Student distribution for \( f \) degrees of freedom, \( t^2 \) falls short of \( t_{\text{max}}^2 \); then we may regard the conclusion as implying that when discussing the value of \( \alpha \) we can make no statement to which attaches a greater fiducial probability than \( P_{\text{max}} \), and which is less vague than was our knowledge before the experimental data became available—that \( \alpha \), if it exists at all, has some value between \( \pm \infty \) and \( + \infty \).

6. For the Cushny-Peebles data, which Sir Ronald Fisher has discussed in this connection (1946, §26.2), we have, if we denote by \( \xi \) and \( \gamma \) the true mean responses to the two drugs,

\[
\begin{align*}
\alpha &= 2.33, \quad \gamma = 0.75, \quad a = \gamma/\alpha = 0.322, \\
v_{xx} &= 0.4009, \quad v_{yy} = 0.3201, \quad v_{xy} = 0.2848, \\
x^2/v_{xx} &= 13.542, \quad \frac{y^2}{v_{yy}} = 1.758, \quad \frac{y v_{xy} - x v_{yy}}{y v_{xx} - x v_{xy}} = 1.466 = A, \\
t_{\text{max}}^2 &= 20.508, \quad P_{\text{max}} = 0.9986, \quad f = 9.
\end{align*}
\]

Fig. 1 shows the curve (6) with these particular numerical values; \( a \), the lowest point of the curve, is at \( (0.322, 0) \), and \( A \), its peak, at \( (1.466, 20.508) \). Its asymptote is the line \( t^2 = x^2/v_{xx} = 13.542 \), corresponding to \( P = 0.99493 \). The curve may be regarded as traced out by the simultaneous motion (see equation 10) of the point \( (\alpha, t^2) \) from \( a \) direct to \( A \), and of the point \( (\alpha_1, t^2) \) from \( a \) through the point \( (0, 1.750) \) towards large negative values of \( \alpha \) and then, after crossing the asymptote at infinity, from large positive values of \( \alpha \) towards \( A \). As indicated in Fig. 1, the values of \( t^2 \) corresponding to probability levels 0.95 and 0.99 lie below, and that corresponding to a level of 0.995 just above, the critical value 13.54. For \( P = 0.999 \), \( t^2 \) exceeds the maximum ordinate of the curve (6), and accordingly, at that level, the fiducial range includes all possible values. The actual limits, and their nature, at these probability levels are:
Fig. 1.—Cushny-Peebles data.
Fiducial limits for the ratio $\alpha = (\text{effect of Drug A})/(\text{effect of drug B}).$

Fig. 2.—Parabola fitted to Goulden's data, and its fiducial quartics.
Fig. 3.—Fiducial regions for the roots \( \xi \) of a quadratic equation (16.1). (real estimates.)

The \( \xi \) scale is horizontal.

Fig. 4.—Fiducial regions for the roots \( \xi \) of a quadratic equation (16.2). (imaginary estimates).

The \( \xi \) scale is horizontal.

Note: the ordinate is taken for convenience as \( \sqrt{\log (1 + t^2/161)} \).
<table>
<thead>
<tr>
<th>P</th>
<th>$\alpha_2$</th>
<th>$\alpha_1$</th>
<th>Nature of Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>+0.657</td>
<td>-0.485</td>
<td>Inclusive</td>
</tr>
<tr>
<td>0.99</td>
<td>+0.808</td>
<td>-2.920</td>
<td>Inclusive</td>
</tr>
<tr>
<td>0.995</td>
<td>+0.896</td>
<td>+146.3</td>
<td>Exclusive</td>
</tr>
<tr>
<td>0.999</td>
<td></td>
<td></td>
<td>Imaginary</td>
</tr>
</tbody>
</table>

7. I regard the curve (6) as an integrated form of the fiducial distribution of $\alpha$. In the example just discussed, for instance, the residual probability $1 - P_{\text{max}} = 0.0014$ attaches to the eventuality that $\alpha$ has no definite finite value, and, remembering that the probabilities quoted in the preceding paragraph are two-tailed ones, we may retabulate the numerical results already obtained in the following form, the first column giving increasingly "remote" values of $\alpha$, the third increasingly remote values of $\alpha_2$, and the central column the fiducial probability, for either root, that $\alpha$ lies between adjacent entries in the relevant column:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Fid. Prob.</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.322</td>
<td>0.475</td>
<td>0.322</td>
</tr>
<tr>
<td>-0.485</td>
<td>0.020</td>
<td>0.657</td>
</tr>
<tr>
<td>-2.920</td>
<td>0.00247</td>
<td>0.808</td>
</tr>
<tr>
<td>$\pm \infty$</td>
<td>0.00003</td>
<td>0.894</td>
</tr>
<tr>
<td>146.3</td>
<td>0.0018</td>
<td>0.896</td>
</tr>
<tr>
<td>1.466</td>
<td></td>
<td>1.466</td>
</tr>
</tbody>
</table>

**Fiducial Intervals for the Roots of a Quadratic Equation**

8. The central curve in Fig. 2 represents a second-order parabola fitted by least-squares to the data listed below, which I quote from Example 10–1 of Goulden’s textbook (1952); $y$ is the mean value of the $n$ observations at abscissa $x$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$n$</th>
<th>$ny$</th>
<th>$x$</th>
<th>$n$</th>
<th>$ny$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>5</td>
<td>-17</td>
<td>1</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>-3</td>
<td>13</td>
<td>-35</td>
<td>2</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td>-2</td>
<td>23</td>
<td>-23</td>
<td>3</td>
<td>26</td>
<td>43</td>
</tr>
<tr>
<td>-1</td>
<td>23</td>
<td>-1</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>35</td>
<td>24</td>
<td>5</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

The residual mean square is $s^2 = 0.8524$ (161 df.), the regression of $y$ on $x$, assumed parabolic, is estimated as

$$Y = b_0 + b_1x + b_2x^2$$  

with

$$b_0 = 0.5677, b_1 = 0.6301, b_2 = -0.1072,$$

and the variance-covariance matrix of the $b$'s is

<table>
<thead>
<tr>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.009734</td>
<td>0.000372</td>
<td>-0.000980</td>
</tr>
<tr>
<td>0.000372</td>
<td>0.001190</td>
<td>-0.000127</td>
</tr>
<tr>
<td>-0.000980</td>
<td>-0.000127</td>
<td>0.000214</td>
</tr>
</tbody>
</table>

The variance of the estimate $Y$, for a given value of $x$, is thus estimated with 161 df as

$$V(Y : x) = 0.00973 + 0.00074x - 0.00077x^2 - 0.00025x^3 + 0.00021x^4,$$
and with this expression for \( V(Y : x) \), and the numerical values of the \( b \)'s given above, the analogue to the Working-Hotelling fiducial hyperbola is the quartic

\[
(y - b_0 - b_1x - b_2x^2)^2 = t^2V(Y : x) 
\]  \hspace{1cm} (15)

The outer curves in Fig. 2 show this quartic for the \( P = 0.95 \) and \( P = 0.99 \) values of \( t \), 1.975 and 2.607.

9. Suppose now that we ask what we can say about the values of \( x \) for which the mean value of \( y \) is equal (i) to zero, (ii) to 1.75. These questions are equivalent, of course, to asking what knowledge we can derive about the roots \( \xi \) of the equations

\[
\beta_0 + \beta_1\xi + \beta_2\xi^2 = 0 \quad . \quad . \quad . \quad (16\cdot1)
\]

\[
\beta_0 + \beta_1\xi + \beta_2\xi^2 = 1.75 \quad . \quad . \quad . \quad (16\cdot2)
\]

from the information that the \( \beta \)'s are estimated as in (11), (12) and (13). Our "point estimates" \( x \) of the values of \( \xi \) will be real in the case of (16\cdot1), imaginary in the case of (16\cdot2), and if we consider the intersections of the lines \( y = 0, y = 1.75 \) with the curves traced in Fig. 2, we appear to arrive at results of the following sort (\( \xi_1 \) is the smaller, \( \xi_2 \) the larger root of the quadratic in question):

<table>
<thead>
<tr>
<th>For equation</th>
<th>( (16\cdot1) )</th>
<th>( (16\cdot2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate of ( \xi_1 ) (( = x_1 ))</td>
<td>( -0.794 )</td>
<td>Imaginary</td>
</tr>
<tr>
<td>( P = 0.95 ) limits for ( \xi_1 )</td>
<td>(-1.02 ) to (-0.55 )</td>
<td>(+3.14 ) to ( ? )</td>
</tr>
<tr>
<td>( P = 0.99 ) limits for ( \xi_1 )</td>
<td>(-1.09 ) to (-0.47 )</td>
<td>(+2.51 ) to ( ? )</td>
</tr>
<tr>
<td>Estimate of ( \xi_2 ) (( = x_2 ))</td>
<td>(+6.672 )</td>
<td>Imaginary</td>
</tr>
<tr>
<td>( P = 0.95 ) limits for ( \xi_2 )</td>
<td>(+5.55 ) to (+8.60 )</td>
<td>(? ) to (+4.21 )</td>
</tr>
<tr>
<td>( P = 0.99 ) limits for ( \xi_2 )</td>
<td>(+5.28 ) to (+9.55 )</td>
<td>(? ) to (+5.66 )</td>
</tr>
</tbody>
</table>

the question-marks representing the imaginary intersections of the line \( y = 1.75 \) with the lower branches of the fiducial quartics.

10. The position does not seem to me to be clarified, if we try to answer the questions of para. 9 not by reference to Fig. 2, but by regarding as a pivotal quantity for a root \( \xi \) of the equation

\[
\beta_0 + \beta_1\xi + \beta_2\xi^2 = Y 
\]  \hspace{1cm} (16)

the ratio

\[
(b_0 - Y + b_1\xi + b_2\xi^2)/\sqrt{V(Y : \xi)} . \quad . \quad . \quad . \quad (17)
\]

From (17) we would deduce immediately that the fiducial range for \( \xi \) is determined by the inequality

\[
(b_0 - Y + b_1\xi + b_2\xi^2)^2 < t^2V(Y : \xi), \quad . \quad . \quad . \quad (18)
\]

but if anything the position has become obscurer, since (18) would appear to supply the same fiducial range for either root of (16). For the roots of (16\cdot2), in particular, we appear to arrive at the limits

\[
P = 0.99 : 2.51 \text{ to } 5.66, \quad P = 0.95 : 3.14 \text{ to } 42.1,
\]

and for a somewhat less stringent significance level (about 0.936) at a fiducial range that reduces to a single point. If the roots of our quadratic were real, as in the case of (16\cdot1), the fiducial range for either of them supplied by (18) would be the sum of the separate ranges obtained by the argument of para. 9.

11. To explain the situation it is necessary, I think, to consider the complete fiducial distributions of \( \xi_1 \) and \( \xi_2 \), or rather, the inverse distributions of the total unit fiducial probability. The discussion of paras. 4 to 7 suggests that in conjunction with the quadratic (16) we may usefully consider the quintic curve \( C \) in the \((\xi, t^2)\) plane represented by the equation

\[
t^2V(Y : \xi) = (b_0 - Y + b_1\xi + b_2\xi^2)^2 . \quad . \quad . \quad . \quad (19)
\]
It has one asymptote, the line \( t^2 = t_0^2 = b_0^2 / V(b_2) = 53.81 \), corresponding to the probability level at which \( b_2 \) ceases to differ significantly from zero. When \( Y = 0 \), the curve \( C_1 \) represented by (19) touches the \( \xi \)-axis at the points \( X_1 \) and \( X_2 \) whose abscissae are the roots of (16.1). As \( \xi \) increases from \(-\infty \) (see Fig. 3), \( C_1 \) rises above its asymptote to a maximum \( t_1^2 \) at a point \( A_1 \), the coordinates of which are about \((-3.32, 215.2)\). The curve then drops to touch the \( \xi \)-axis at \( X_1 \), rises again to a second maximum \( t_2^2 \) at a point \( A_2 \), (2.14, 207.5) roughly, drops again to touch the \( \xi \)-axis at \( X_2 \) and finally, as \( \xi \) tends to \(+\infty \), approaches its asymptote again from below. Any parallel to the \( \xi \)-axis cuts the curve in four points, which will all be real as long as \( t^2 \) lies in the range \( 0 < t^2 < t_0^2 \). In the range \( t_0^2 < t^2 < t_1^2 \) only two of the intersections will be real, and for \( t^2 > t_1^2 \) none of them will. When \( Y = 1.75 \) (and more generally, when our estimates of the roots of (16) are imaginary) the curve \( C_2 \) represented by (19) has no real intersections with the \( \xi \)-axis, (see Fig. 4), and has only a single maximum (\( A \)) and only a single minimum (\( X \), say). Its shape is thus much the same as that of the cubic of Fig. 1. In the present example, the co-ordinates of \( X \) and \( A \) are approximately \((3.62, 3.47)\) and \(-2.18, 789.3)\), corresponding to significance levels \( P_a = 0.936 \) and \( P_a = 1 - (0.59) 10^{-65} \) approximately.

12. In the light of Fig. 3, I think that question (i) of para. 9 is appropriately answered by saying that if \( L \) is the line lying parallel to the \( \xi \)-axis at a distance \( t^2 \) above it, then

(a) for \( t^2 < t_1^2 \) the corresponding fiducial range for \( \xi_1 \) is the portion of \( L \) lying between the segments \( A_1 X_1 \) and \( X_1 A_2 \) of the curve \( C_1 \) of Fig. 3, and that for \( \xi_2 \) is the remaining portion of \( L \) lying above \( C_1 \). If \( t^2 > t_0^2 \), the fiducial range for \( \xi_2 \) will include indefinitely large positive and negative values of \( \xi \):

(b) at the significance level \( P_a = 1 - (0.93) 10^{-30} \) corresponding to \( t_2 \), the data cease to discriminate between \( \xi_1 \) and \( \xi_2 \); in other words, the quadratic discriminant ceases at that level to exceed zero significantly:

(c) for \( t_1^2 < t^2 < t_2^2 \), the portion of \( L \) lying above \( C_1 \) is the fiducial range common to \( \xi_1 \) and \( \xi_2 \):

(d) at the significance level \( P_1 \) corresponding to \( t_1 \), the fiducial range common to \( \xi_1 \) and \( \xi_2 \) includes the whole continuum, and the residual probability \( 1 - P_1 = (0.17) 10^{-30} \) attaches to the eventuality that \( \xi_1 \) and \( \xi_2 \) have no definite value.

The tangent at \( A_2 \) to \( C_1 \) may thus be regarded as dividing the area lying above \( C_1 \) into the fiducial regions for \( \xi_1 \), for \( \xi_2 \) and for \( \xi_1 \) and \( \xi_2 \) shown in Fig. 3.

13. In the light of Fig. 4, similarly, I think that question (ii) of para. 9 is appropriately answered by saying that

(a) there is a fiducial probability \( P_a = 0.936 \) that the values of \( \xi \) sought do not exist:

(b) the quadratic discriminant \( b_1^2 - 4b_0b_a \) ceases to fall significantly short of zero at the probability level \( P_a \):

(c) the statement that can be made at a significance level \( P \) lying between \( P_a \) and \( P_0 \) is that there is a fiducial probability \( P \) that the values of \( \xi \) sought do not exist, and an additional probability \( P - P_a \) attaching to the range defined by the portion above the curve \( C_2 \) of the appropriate parallel to the \( \xi \)-axis:

(d) this range includes indefinitely large values, positive and negative, whenever \( t^2 \) exceeds \( t_2^2 \) and extends to all possible values when \( t^2 \) reaches \( t_1^2 \). The residual probability \( 1 - P_a \) attaches to the eventuality that the quantities \( \xi \) sought have no definite finite value.

In a case such as this, it would seem, we cannot expect, at any significance level, to find different fiducial ranges for the two roots.

14. I have not thought it worth while to discuss in detail the exceptional case, in which our estimates of \( \xi_1 \) and \( \xi_2 \) happen to coincide. In such a case again, there can be no possibility, at any level, of distinguishing different ranges for \( \xi_1 \) and \( \xi_2 \), and their common range will presumably be defined by a curve similar in appearance to that of Fig. 1. Although our estimate of the common value may be regarded as provided by the root of a simple equation, the different assumptions concerning the distributions of the coefficients prevent this case from reducing to a study identical with that of paras. 2 to 7. Also, I have not investigated the complete variety of forms that the
curve C might assume. If in my example \( t^2 \) had happened to exceed \( t^2 \), I would have regarded the tangent to \( C_1 \) at \( A_1 \), instead of that at \( A_2 \), as defining the various fiducial regions for \( \xi_1 \) and \( \xi_2 \).

**Some General Remarks**

15. In many schools of mathematical statistics it is the current fashion to discuss problems of interval estimation in terms of “confidence coefficients” rather than of “fiducial probability”; and I know that the phraseology that I have used in the body of this paper will not be universally popular. I have tried throughout, however, to write in accordance with established practice, and in particular with the concise definitions given in the glossary of a well-known textbook (Lyle, 1942):

- **Fiducial Limits.** Limits within which we may expect the true or ‘population’ value of an estimate to lie with a given Fiducial Probability.
- **Fiducial Probability.** That probability, or level of odds, decided upon for use in tests of significance of the differences between a statistic and all possible parameters of which it may be deemed to be an estimate.
- **Fiducial Probability, Statements of.** Probability statements relating to unknown parameters based on tests of significance only, in contrast with statements of so-called ‘inverse probability’ which are based on a postulated distribution a priori.”

By way of further illustration, I quote the concluding sentences of Mr. Lyle’s account of the limits for a regression coefficient (loc. cit., p. 54):

“These limits are called Fiducial Limits, and when they are calculated for the 5 per cent. level we can say that the probability that the true or population value lies within these limits is 95 per cent. or 0·95 and the probability that it lies outside these limits is 5 per cent. or 0·05.

This probability is called a Fiducial Probability because it measures the degree of confidence we can place in the statement that the population of which our data are conceived to be a sample has a value of \( B \) lying within those limits.”

16. The earlier discussions of the ratio problem (Irwin, 1943, and Fieller, 1944) were both phrased in terms of “fiducial ranges”, but the arguments by which the results of those papers were established were in fact straightforward “confidence interval” ones. Similarly, it may easily be seen that the ranges supplied by (18) are “confidence intervals” in the sense in which Professor Neyman has defined that term. I do not think that this fact in itself justifies us in regarding (18), or the obvious analogue to it in the case of the general equation (1), as providing a complete solution to my problems. It seems to me that one of the important differences between the “fiducial” and “confidence” theories of interval estimation is, that the former leads us naturally to consider the conclusions established, at a succession of different significance levels, by the same body of data; whereas in the latter theory the natural succession to consider is the series of statements that would be made, with a fixed confidence coefficient, on the basis of a hypothetical series of different sets of data. In “confidence interval” theory there seems no room for explanatory statements of the form of para. 13(c); and in the context of the problems that I have been discussing, I feel that it is the “fiducial” outlook that is needed, if we are to escape from apparent paradoxes such as those to which I have referred in paragraph 10.

**Acknowledgments**

I have profited from discussions with Professor G. A. Barnard, and from the opportunity to read (and disagree with) the draft of the paper that Miss Creasy is about to present. I am indebted to Miss E. Ducker for her help in the preparation of this paper, and to the Chief Scientist, Ministry of Supply, for permission to deliver it.

**Appendix**

A.1. I have been concerned in this paper to answer, in some detail, the questions posed by equations (2) and (16). Similarly, we might hope to tackle the general problem of para. 1 by studying the curve represented in the \((x, r^2)\) plane by the equation

\[
(F(b, x))^2 = r^2 V(F(b, x)) = r^2 \sum_{ij} F_i(x) F_j(x) \quad \text{...} \quad (20)
\]
I do not wish to leave the impression, however, that I consider such a detailed study always necessary in practice. In the analysis of large-scale biological assays it frequently suffices to assess the reliability of the estimate of relative potency by the well-known “approximate standard error”, and dispense with the accurate assessment supplied by (5). The arithmetical labour saved is in this case trivial; but in more complicated cases we should clearly hope that a study of (20) could be replaced by some simpler device. An “approximate variance” for our estimate \( a \) could normally be derived by squaring and averaging the “statistical differential” equation

\[
\delta a \Sigma b_r F_r'(a) = - \Sigma F_r(a) \delta b_r, \quad \ldots \quad \ldots \quad \ldots \quad (21)
\]

or by some effectively equivalent process.

Fox, Flory and Bueche (1951), for instance, have suggested that the osmotic pressure \( p \) of a polymer solution is related to its concentration \( C \) by an equation of the form

\[
p = \alpha C + \alpha \beta C^2 + 5 \alpha \beta C^3/8 \quad \ldots \quad \ldots \quad \ldots \quad (22)
\]

where \( \alpha \) is the molecular weight of the polymer. If we accept this theory we should thus be able to derive an estimate of \( \alpha \) from a series of \((p, C)\) determinations; the maximum-likelihood estimates of \( \alpha \) and \( \beta \) may be obtained by minimizing the sum of squared residuals corresponding to (22), and therefore satisfy the equations

\[
a \Sigma ((C^2 + 5bC^3) (C + bC^2 + 5b^2C^3/8)) = \Sigma (pC^2 + 5bC^3/4) \quad \ldots \quad \ldots \quad \ldots
\]

\[
a \Sigma ((2C + bC^2) (C + bC^2 + 5b^2C^3/8)) = \Sigma (p(2C + bC^3)) \quad \ldots \quad \ldots \quad \ldots
\]

We could, if it were essential, arrive at exact fiducial limits for \( \alpha \) by first eliminating \( a \) from these equations to derive a quartic yielding \( b \), and then computing limits for \( \alpha \) from those for \( \beta \). In practice, however (Ducker et al., 1952), it has proved sufficiently accurate to rely on the usual maximum likelihood expressions for \( V(a) \) and \( V(b) \) derived from the sum of squared residuals.

A.2. One of the referees has suggested that I should record some examples that lead me to reject any argument producing different fiducial limits for a ratio from those supplied by equation (5). I should mention that Miss Creasy replies to these examples in her para. 3(c).

Suppose that \( L \) is the estimate of a true regression line \( \lambda \) obtained by fitting a straight line in the usual way to \( n \) points \((x_r, y_r)\), that \( \bar{x}, \bar{y}, (x^2), (xy), (y^2) \) are the means and reduced sums of squares and products of the \((x_r, y_r)\), and that \( b = (xy)/(x^2) \). We might ask what is logically the same question about \( \lambda \) in three different ways:

(i) does \( \lambda \) pass through the origin 0?
(ii) is the intercept of \( \lambda \) on the \( y \)-axis zero?
(iii) is the intercept of \( \lambda \) on the \( x \)-axis zero?

Because these questions are worded differently, we may well use different routes to arrive at the statistical techniques by which we shall answer them. Because the questions are logically identical, the answers that we give to them, when working at any particular significance level, ought in my view to be either all “Yes” or else all “No”. If this were not the case, we would be equipping ourselves with a battery of methods liable to land us in self-contradictions.

It seems to me that the natural approaches to the three questions are these \((s^2 \text{ with } n - 2 \text{ df} \text{ is the residual mean square about } L)\):

(i) the residual sum of squares about \( L \) falls short of that about a line fitted to the \((x_r, y_r)\) and constrained to pass through 0 by \((\bar{y} - b \bar{x})^2/(1/n + \bar{x}^2/(x^2))\); thus we answer “No” to (i) unless

\[
(\bar{y} - b \bar{x})^2 < n s^2(1/n + \bar{x}^2/(x^2)) \quad \ldots \quad \ldots \quad \ldots \quad (23)
\]

(ii) the estimated variance of the intercept \((\bar{y} - b \bar{x})\) of \( L \) on the \( y \)-axis is \( s^2(1/n + \bar{x}^2/(x^2)) \).

We shall thus answer “No” to (ii) unless (23) is satisfied;

(iii) the fiducial range for the intercept \( x_0 = (b \bar{x} - \bar{y})/b \) of \( L \) on the \( x \)-axis is, from (5), determined by

\[
(b^2 - n s^2/2) x_0^2 - 2(b(b \bar{x} - \bar{y}) - n s^2 \bar{x}^2/(x^2)) x_0 + [(b \bar{x} - \bar{y})^2 - n s^2 \bar{x}^2/(x^2) + 1/n] < 0 \quad \ldots \quad \ldots \quad \ldots \quad (24)
\]
and includes zero only when (23) is satisfied. Any alternative to (5) giving a different independent term from that in (24) would lead to contradiction.

More generally we might ask two further questions about $\lambda$, which seem to me to be logically identical with each other:

(iv) is the intercept of $\lambda$ on the $x$-axis equal to $x_0$?

(v) does $\lambda$ pass through the point $(x_0, 0)$?

In my view (iv) is to be answered by "No" unless (24) is satisfied. To answer (v), we might note that the residual sum of squares about $L$ falls short of that about a fitted line constrained to pass through $(x_0, 0)$ by

$$\{(\bar{y} - b\bar{x}) + bx_0\}^2 / (1/n + (\bar{x} - x_0)^2 / (\sigma^2)),$$

which is judged insignificant when, and only when, (24) is satisfied.

One might produce other examples to illustrate the equivalence of (5) and standard least-squares theory—e.g., in slope-ratio assays, as well as in those in which the log (dose)-response line is assumed to be straight, we can identify the fiducial range supplied by (5) with that resulting from considering the variation permissible in the "between preparations" degree of freedom. That it is possible to construct such examples results, of course, from the fact that (5) is formally identical with an ordinary variance-ratio test in which the numerator has one degree of freedom.

References