Since we are interested in just $T(z)$, $F$ is the identity function, i.e., $F(x) = x$.

So, $F'(x) = 1$. Let $u = T(z)$ \Rightarrow $G(u) = e^u$. Then the coefficients $t_n$ satisfy

$$[z^n]T(z) = \frac{1}{n} \left[ u^{n-1} \right] e^{nu}$$

Generating function of $e^{nu}$ equals

$$\sum_{n \geq 0} \frac{n^n}{n!} u^n$$

It follows that

$$\frac{1}{n} \left[ u^{n-1} \right] e^{nu} = \frac{n^{n-1}}{n!}$$

(a) In Pentagon example, we’ve basically witnessed that for each $m$ elements in $p_n$,

$$p_{nm} = p_0 I_{\{n=0\}} + \sum_i P(X_n = m \mid X_{n-1} = i)P(X_{n-1} = i)$$

Therefore,

$$p_n = p_0 I_{\{n=0\}} + p_{n-1} P$$

Plug in the above value of $p_n$ into the definition of generating function $G(z)$, i.e.

$$G(z) = \sum_n p_n z^n$$

and the result follows after a little algebra.

(b) This is very similar to the one-dimensional case, $\frac{1}{1-pz}$. So one guess would be

$$\sum_{k=0}^{\infty} p^k z^k$$

In the general case, given an invertible matrix $P$ and scalar $a$,

$$(I \pm aP)^{-1} = I + \sum_{k=1}^{\infty} (\mp 1)^k P^k a^k$$
Let \( I = P^0z^0 \) and substitute \( a \) with \( z \). Our guess is confirmed.

(c) It is given by the coefficients of \( p_0(I - zP)^{-1} \), which is \( p_0P^n \).

### 3 Double Roots

In double roots case, we end up with a system of equations that has one equation but two unknowns. One alternative solution involves using the fact that the derivative of a polynomial of order 2 evaluates to zero if its root has multiplicity 2. Using this fact, we can differentiate

\[
G(z)(az^2 + bz + c) = zP(z; g_1, g_{m-1})
\]

and evaluate it at its root to get a second equation.

If one of the roots is zero, then \( c = 0 \). So \( cg_1 = 0 \) and as a result, we only have to solve for \( g_{m-1} \). When both are zero, then \( b = c = 0 \). The recurrence equations become

\[
ag_{k-1} = u_k
\]

\[
g_0 = u_0
\]

\[
g_m = u_m
\]

Using boundary conditions, we can determine all of the \( g_k \) recursively. Therefore, we don’t have anything to solve! Not very interesting...

### 4 Poisson Naturally

(a) First divide \((0, t]\) into \( n \) intervals of length \( t/n \).

Next, let \( \epsilon_i = 1 \) if 1 event occurs on the interval \(((i-1)t/n, it/n]\), and 0 otherwise.

So, \( S_n = \epsilon_1 + \ldots + \epsilon_n \sim \text{Bin}(n,p) \) where \( p = \frac{\lambda t}{n} + o(t/n) \).

Using the Poisson approximation of Binomial probabilities,

\[
P(S_n = k) = \frac{e^{-np}(np)^k}{k!} = \frac{e^{-\lambda t - tz}(\lambda t + tz)^k}{k!} \quad \text{note } z = \frac{o(t/n)}{t/n}
\]
\[ e^{-\lambda t} (\lambda t)^k \frac{1}{k!} \quad \text{as } n \to \infty \]

(b) \( P(S_1 \leq s \mid N_t = 1) \)

\[
= \frac{P(N_s = 1 \mid N_t = 1)}{P(N_t = 1)} \\
= \frac{P(N_s = 1 \cap N_t = 1)}{P(N_t = 1)} \\
= \frac{P(N_t = 1 \mid N_s = 1)P(N_s = 1)}{P(N_t = 1)} \\
= \frac{s}{t} \quad \text{for } s < t
\]

This is the cdf of Uniform(0, t).

(c) We’ll solve the general case in part (d). The argument is essentially the same.

(d) For simplicity, let \( s_0 = 0 \). For \( 0 \leq s_1 \leq s_2 \ldots \leq s_n \leq t \),

\[
P(S_1 \leq s_1, \ldots, S_n \leq s_n \mid N_t = n)
\]

\[
= \frac{P(S_1 \leq s_1, \ldots, S_n \leq s_n, N_t = n)}{P(N_t = n)} \\
= \frac{P(N_t - N_{s_n} = 0)P(N_{s_n} - N_{s_{n-1}} = 1)\ldots P(N_{s_1} - N_{s_0} = 1)}{P(N_t = n)} \\
= \frac{\exp\{-\lambda(t - s_n)\}}{\left(\frac{\lambda e^{-\lambda t}}{n!}\right)} \left(\prod_{j=1}^{n} \exp\{-\lambda(s_j - s_{j-1})\}\lambda(s_j - s_{j-1})\right) \\
= \frac{\lambda^n e^{-\lambda t} \prod_{j=1}^{n} (s_j - s_{j-1})}{\left(\frac{\lambda e^{-\lambda t}}{n!}\right)} \\
= \frac{n!}{t^n} \prod_{j=1}^{n} (s_j - s_{j-1})
\]

Take derivative with respect to \( s_1, \ldots, s_n \) to get the joint pdf.

The resulting conditional joint pdf is \( \frac{n!}{t^n} \), for \( t > 0 \)

This is the pdf of the order statistics for a random sample of \( n \) Uniform(0, t) random variables.
Let $N(t)$, $N_1(t)$ and $N_2(t)$ denote the number of customers entering the building, "Good Eats", and "Eat Good" in $[0, t]$. The claim is $N_1(t)$ is a Poisson process with rate $p\lambda$ and $N_2(t)$ is a Poisson process with rate $q\lambda$. Using law of total probability,

$$P(N_1(t) = k) = \sum_{i=0}^{\infty} P(N_1(t) = k | N(t) = i)P(N(t) = i)$$

$$= \sum_{i} \binom{i}{k} p^k q^{i-k} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$$

$$= \frac{(p\lambda t)^k e^{-\lambda t}}{k!} \sum_{i} \frac{[(1 - p)\lambda t]^{i-k}}{(i-k)!}$$

$$= \frac{(p\lambda t)^k e^{-\lambda t}}{k!} e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k e^{-p\lambda t}}{k!}$$

We can do the same for $N_2(t)$.

(a) Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $\mathcal{G}_n = \sigma(Y_0, \ldots, Y_n)$

$$E(X_{n+1} | \mathcal{F}_n) = E[E(Z | \mathcal{G}_{n+1}) | \mathcal{F}_n]$$

$$= E(E[ E(Z | \mathcal{G}_{n+1}) | \mathcal{G}_n] | \mathcal{F}_n)$$

$$= E[E(Z | \mathcal{G}_n) | \mathcal{F}_n]$$

$$= E[X_n | \mathcal{F}_n] = X_n$$

Second equality holds because $\mathcal{F}_n \subset \mathcal{G}_n$. The subset relationship is due to the fact that $X_n$ is a function of the random variables in $\mathcal{G}_n$. Third equality holds by the Mighty Conditioning Identity.

(b) $E(X_{n+1} | \mathcal{F}_n) = E\left( X_n \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} | \mathcal{F}_n \right)$

$$= X_n E\left( \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} | \mathcal{F}_n \right)$$
\[ \begin{align*}
= & \quad X_n E \left( \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} \right) \\
= & \quad X_n \int \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} f_0(Y_{n+1}) dY_{n+1} \\
= & \quad X_n \\
\end{align*} \]

Second and third equalities follow from Enhanced Scaling Rule and i.i.d. of \( Y_k \)'s respectively.

(c)

\[ \begin{align*}
E(X_{n+1} | \mathcal{F}_n) &= \frac{1}{n+3} \cdot E \left[ Y_{n+1} \bigg| \frac{Y_0}{2}, \ldots, \frac{Y_n}{n+2} \right] \\
&= \frac{1}{n+3} \left[ Y_n + \frac{Y_{n+1}}{n+2} \right] \\
&= \frac{Y_n}{n+2} = X_n \\
\end{align*} \]

Lemma Time

\[ \begin{align*}
\star : & \quad S_k(t) \geq 0 \\
\circ : & \quad S_k(t) \text{ is only nonzero for one value of } k \text{ and zero everywhere else} \\
\sup_t |f_n(t) - f(t)| &= \sup_t \left| \sum_{k=n+1}^{\infty} a_k S_k(t) \right| \\
&\leq \sup_t \sum_{\ell=1}^{\infty} \sum_{k=2^\ell}^{2^{\ell+1}-1} |a_k| S_k(t) \quad \text{where } \ell < \log_2(n+1) \quad \text{(by } \star) \\
&\leq \sup_t \sum_{\ell=1}^{\infty} \sum_{k=2^\ell}^{2^{\ell+1}-1} M(2^{\ell})^\gamma S_k(t) \quad \text{(since } k \text{ is at most } 2^\ell) \\
&= \sup_t \sum_{\ell=1}^{\infty} M(2^{\ell})^\gamma \sum_{k=2^\ell}^{2^{\ell+1}-1} S_k(t) \\
&\leq \sup_t \sum_{\ell=1}^{\infty} M(2^{\ell})^\gamma \left[ 2^{-\frac{1}{2}(\ell+1)} \right] \\
&\quad \text{by } \circ \text{ and that the value is } [2^{-\frac{1}{2}(\ell+1)}] \\
\end{align*} \]
\[ = \sup \sum_{\ell=i}^{\infty} M' \cdot 2^{\ell(\gamma - 1/2)} \]

\[ \to 0 \quad \text{as } n \to \infty \text{ since } 0 \leq \gamma < 1/2 \]