Plan Fun with White Noise, Last Part

1. The Poisson Process cont’d/revisited
3. Brownian Motion
3. Stationary Processes (if time allows – ha ha – or skip til later)

Next Time: Markov Processes (finally)

Reading: G&S 6.1, 6.2, generating function sheet
Homework 2 due Today
Homework 3 due next week

Figure 1.

Example 2. Poisson Process
Consider a random walk with $S_0 = 0$ and $X_i \overset{iid}{\sim} \text{Exponential}(\lambda)$. (Note: If $W_i = X_i - \lambda$, $S_n = \sum_i W_i + n\lambda$, a sum of white noise plus drift.)

Then, $S_n$ has a Gamma($n$, $\lambda$) distribution.

Define $N_t = \max \{n \geq 0 \mid S_n \leq t\}$.

A few properties directly from the definition:

1. $N_0 = 0$.
2. $N_t \geq n \iff S_n \leq t$.
3. $N_t = n \iff S_n \leq t$ and $S_{n+1} > t$.
4. If $s < t$, $N_t - N_s$ counts the number of “arrivals” between $s$ and $t$. This random variable $N_t - N_s$ is often called the increment of the process over $(s,t)$.

We have the following.

\[
\begin{align*}
\mathbb{P}\{N_t = k\} &= \mathbb{P}\{S_k \leq t, S_{k+1} > t\} \\
&= \mathbb{E}\mathbb{P}\{S_k \leq t, S_{k+1} > t \mid S_k\} \\
&= \int_0^t \mathbb{P}\{X_{k+1} > t - u\} \mathbb{P}\{S_k \text{ near } u\} \\
&= \int_0^t e^{-\lambda(t-u)} \frac{\lambda^k u^{k-1}}{k!} e^{-\lambda u} du \\
&= e^{-\lambda t} \frac{\lambda^k}{k!} \int_0^t u^{k-1} du \\
&= e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
\end{align*}
\]
so $N_t$ has a Poisson($\lambda t$) distribution.

Next, we consider the distribution of $N_t - N_s$ for some $s < t$.

\[
\mathbb{P}\{N_t - N_s \geq k\}
= \mathbb{P}\{S_{k+N_s} \leq t\} 
= \mathbb{P}\{S_{k+N_s} - S_{N_s} \leq t - S_{N_s}\} 
= \mathbb{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s})\} 
= \mathbb{E}\mathbb{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s}) \mid N_s, S_{N_s}\} 
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s}) \mid N_s = n, S_{N_s} \text{ near } u\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{7}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{S_{k+n} - S_n \leq t - s + (s - u) \mid N_s = n, S_{N_s} \text{ near } u\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{8}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{X_{n+1} - (s - u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \mid N_s = n, S_{N_s} \text{ near } u\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{9}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{X_{n+1} - (s - u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \mid X_{n+1} > s - u, S_n \text{ near } u\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{10}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{X_{n+1} - (s - u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \mid X_{n+1} > s - u\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{11}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{X_{n+1} + \sum_{i=n+2}^{n+k} X_i \leq t - s\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{12}
\]

\[
= \sum_{n=0}^{\infty} \int_0^s \mathbb{P}\{S_{n+k} - S_n \leq t - s\} \mathbb{P}\{N_s = n, S_{N_s} \text{ near } u\} \, ds \tag{13}
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}\{S_{n+k} - S_n \leq t - s\} \mathbb{P}\{N_s = n\} \tag{14}
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}\{S_k \leq t - s\} \mathbb{P}\{N_s = n\} \tag{15}
\]

\[
= \sum_{n=0}^{\infty} \mathbb{P}\{N_t - N_s \geq k\} \tag{16}
\]

That step from (16) to (17) requires an argument. It follows from the relation

\[
\mathbb{P}\{X > t + u \mid X > u\} = \mathbb{P}\{X > t\} \tag{17}
\]

for a random variable $X$ with an Exponential distribution. Thus we see that $N_t - N_s$ has the same distribution as $N_{t-s}$.
Now consider \( s' < t' < s < t \). We could use the same basic argument to show that \( N_t - N_s \) and \( N_{t'} - N_{s'} \) are independent but it gets a bit messy. It’s easier using generating functions. Note that

\[
G(z; \lambda) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} z^n = e^{\lambda(z-1)}. \tag{23}
\]

From this, we get the \( G \) of \( N_t \):

\[
G_{N_t}(z) = e^{\lambda(t-1)} \tag{24}
\]

\[
= e^{\lambda(t-s+s-t'+t'-s'+s'-0)(z-1)} \tag{25}
\]

\[
= e^{\lambda(t-s)(z-1)} e^{\lambda(s-t')(z-1)} e^{\lambda(t'-s')(z-1)} e^{\lambda(s'-0)(z-1)} \tag{26}
\]

\[
= G_{N_{t-s}}(z) G_{N_{s-t'}}(z) G_{N_{t'-s'}}(z) G_{N_{s'-0}}(z) \tag{27}
\]

\[
= G_{N_{t-N_s}}(z) G_{N_{s-N_{t'}}}(z) G_{N_{t'-N_{s'}}}(z) G_{N_{s'-N_0}}(z) \tag{28}
\]

But \( N_t = N_t - N_s + N_s - N_{t'} + N_{t'} - N_{s'} + N_{s'} - N_0 \), and this equality of generating functions implies that the components are independent. (There’s a brief argument needed, which I’ll show you, to make this rigorous.) Conversely, if we show independence first, we could show equality in distribution with the same relation.

Thus, from a white noise random walk, we get the following process.

**Process 3.** Let \( (N_t)_{t \geq 0} \) be a process with the following properties:

1. \( N_0 = 0 \)
2. For \( 0 \leq s < t \), \( N_t - N_s \) has a Poisson \((\lambda(t-s))\) distribution.
3. For \( 0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m \), the random variables \( N_{t_i} - N_{s_i} \) are independent.

This is called a (homogeneous) **Poisson process** with rate \( \lambda \).

Property 2 has two parts. The first is the specific distribution of the increment \( N_t - N_s \). The second is that the distribution of the increment depends on time only through \( t - s \). This is the property of **stationary increments**: the distributions of increments in two time intervals of the same length are equal.

Property 3 is called **independent increments**: the increments in disjoint time intervals are stochastically independent.

Poisson processes are examples of both **renewal processes** and **point processes**, both of which we will study later.
Definition 4. Let $L^2(0,1)$ be (up to some formal details) the set of functions $g$ on $(0,1)$ such that $\int_0^1 g^2 < \infty$.

A complete, orthonormal basis $(\psi_n)$ for $L^2(0,1)$ is a countable subset of $L^2(0,1)$ such that $\int \psi_j \psi_k = \delta_{jk}$ and for any $g \in L^2(0,1)$, we can find $c_n = \int g \psi_n$ such that

$$\int \left( g - \sum_{k=0}^n c_k \psi_k \right)^2 \to 0,$$

as $n \to \infty$.

Example 5. Brownian Motion

Define a process $(\xi_t)_{t \geq 0}$ as follows. Let $(A_n)_{n \geq 0}$ be a standard normal white noise Process, i.e., $A_n$ are iid Normal(0,1). Define

$$\xi_t(\omega) = \sum_{n=0}^{\infty} A_n(\omega) \psi_n(t),$$

for a particular complete orthonormal basis $(\psi)$.

Taking some liberties (we’ll see a more formal derivation another time), we will think of $\xi_t$ as a continuous white noise process meaning that in some formal sense we should have $\mathbb{E} \xi_t = 0$ and $\mathbb{E} \xi_s \xi_t = \delta(s-t)$. Arguing loosely, this works:

$$\mathbb{E} \xi_t = \sum_n \mathbb{E} A_n \psi_n(t) = 0$$

(31)

$$\mathbb{E} \xi_s \xi_t = \sum_{n,m} \mathbb{E} A_n A_m \psi_n(t) \psi_m(s) = \sum_n \psi_n(t) \psi_n(s),$$

(32)

(33)

which “makes sense” when $s \neq t$ because

$$\int dt \mathbb{E} \xi_s \xi_t = \sum_n \psi_n(s) \int dt \psi_n(t) = 0.$$  

(34)

But don’t take that last calculation too seriously.

Now, just as we got a random walk process by taking cumulative sums of a discrete white noise process, we can see what we get when we take cumulative integrals of a continuous white noise process.

Define

$$W_t = \int_0^t \xi_s ds = \sum_{n=0}^{\infty} A_n \int_0^t \psi_n(s) ds,$$

(35)

where we choose a specific basis $\psi_n$.

Definition 6. Define $H = 1_{(0,1/2]} - 1_{(1/2,1]}$. Then let

$$H_{jk}(t) = 2^{j/2} H(2^j t - k).$$

Then, $H_0 = 1_{(0,1]}$ and $H_{jk}$ for $j \geq 0$, $k = 0, \ldots, 2^j - 1$ form a complete orthonormal basis for $L^2(0,1)$. It is called the Haar basis.

To see this note that $\int H_0^2 = 1$, $\int H_{jk} H_{j'k'} = \delta_{jj'} \delta_{kk'}$, and $\int H_{jk} = 0$. And we have that $\alpha H_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{jk} H_{jk}$ gives a representation for all piecewise constant functions on dyadic intervals of length $2^{-J}$. (We’ll have a cool martingale proof of this another time.)
Example 5 cont’d Now order the Haar functions \((H_0, H_{00}, H_{10}, H_{11}, H_{20}, H_{21}, H_{22}, H_{23}, \ldots)\) and label these as \(\psi_n\) for \(n \geq 0\). (For \(2^j \leq n < 2^{j+1}\) and \(j \in \mathbb{Z}_+\), take \(k = n - 2^j\) and let \(H_n \equiv H_{jk}\).)

For \(n \geq 1\),

\[
\int_0^t H_n(s) \, ds \equiv S_n(t),
\]

called the Schauder function.

It follows that

\[
W_t = \sum_{n \geq 0} A_n S_n(t).
\]

Figure 7.

\[
2^{-j/2-1} \quad S_n(t) \quad 2^j \leq n < 2^{j+1} \quad k = n - 2^j
\]

\[
k2^{-j} \quad (k + 1)2^{-j}
\]

Lemma 8. Let \((a_k)\) be a real sequence that satisfies \(|a_k| = O(k^\gamma)\) for some \(0 \leq \gamma < 1/2\). Define \(f(t) = \sum_{k \geq 0} a_k S_k(t)\) and \(f_n(t)\) be the corresponding partial sum. Then \(f_n \to f\) uniformly on \((0, 1)\), meaning that \(\sup_{0 < t < 1} |f_n(t) - f(t)| \to 0\).

Lemma 9. A standard normal white noise sequence \(A_n\) satisfies \(|A_n| = O(\sqrt{\log n})\) with probability 1.

Lemma 10. If \(0 \leq s, t \leq 1\),

\[
\sum_{n \geq 0} S_n(s) S_n(t) = \min(s, t).
\]

Proof of Lemma 10 Let \(\phi_s = 1_{[0, s]}\). Then, if \(s \leq t\),

\[
s = \int_0^1 \phi_t \phi_s = \sum_{n \geq 0} a_n b_n
\]

where

\[
a_n = \int_0^1 \phi_t H_n = S_n(t)
\]

\[
b_n = \int_0^1 \phi_s H_n = S_n(s).
\]
Example 5 cont'd So, $W_t$ as defined exists as a random function. We get the following:

\[ EW_t = \sum_{n \geq 0} E A_n S_n(t) = 0 \]  \hspace{1cm} (43)

\[ EW_t^2 = \sum_{n \geq 0} E A_n^2 S_n(t) = t \]  \hspace{1cm} (44)

\[ EW_sW_t = \sum_{n \geq 0} E A_n A_m S_n(t)S_m(s) = \min(s, t) \]  \hspace{1cm} (45)

and for $u < s < t$,

\[ E(W_t - W_s)W_u = \sum_{n,m \geq 0} E A_n A_m (S_n(t) - S_n(s))S_m(u) = \min(t, u) - \min(s, u) = 0. \]  \hspace{1cm} (46)

Moreover, using the characteristic generating functions with $s \leq t$,

\[ E e^{i\lambda(W_t - W_s)} = E e^{i\lambda \sum_n A_n (S_n(t) - S_n(s))} \]
\[ = \prod_{n=0}^{\infty} E e^{i\lambda A_n (S_n(t) - S_n(s))} \]
\[ = \prod_{n=0}^{\infty} e^{-\frac{\lambda^2}{2}(S_n(t) - S_n(s))^2} \]
\[ = e^{-\frac{\lambda^2}{2}(t-2s+s)} \]
\[ = e^{-\frac{\lambda^2}{2}(t-s)}, \]

using normality of the $A_n$s. Hence, $W_t - W_s$ has a $\text{Normal}(0, t - s)$ distribution. We get the following process.

Process 11. Thus, derived from a white noise process, we get a process $(W_t)_{t \geq 0}$ with the following properties:

1. $W_0 = 0$
2. For $0 \leq s < t$, $W_t - W_s$ has a $\text{Normal}(0, t - s)$ distribution.
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots$, the random variables $W_{t_i} - W_{s_i}$ are independent.

We call this process a Weiner process or equivalently, a Brownian Motion.
**Definition 12.** A stochastic process $X = (X_t)$ is called *strongly stationary* if for any $n \geq 1$, any $t_1, \ldots, t_n$, and any $h$, the two vectors

$$(X_{t_1}, \ldots, X_{t_n}) \quad \text{and} \quad (X_{t_1+h}, \ldots, X_{t_n+h})$$

have the same distribution.

**Definition 13.** A stochastic process $X = (X_t)$ is called *weakly stationary* (aka *second-order stationary*) if for any $t_1, t_2$ and $h$,

$$\text{EX}_{t_1} = \text{EX}_{t_2}$$

$$\text{Cov}(X_{t_1}, X_{t_2}) = \text{Cov}(X_{t_1+h}, X_{t_2+h}).$$

**Examples 14.**

1. Is the white noise process stationary? In which sense?
2. What about the Poisson process? Weiner Process?
3. ARMA processes

**Definition 15.** If $X$ is a (weakly) stationary process, then three important functions that characterize the process:

1. The *autocovariance function* $R$ is defined by

$$R(t) = \text{Cov}(X_0, X_t).$$

2. The *autocorrelation function* $\rho$ is defined by

$$\rho(t) = \text{Cor}(X_0, X_t) = \frac{R(t)}{R(0)}.$$  

3. The *spectral density* is defined when $\rho$ has the representation

$$\rho(t) = \int e^{it\lambda}dF(\lambda)$$

for some distribution function $F$. The spectral density is the function $f(\lambda) = F'(\lambda)$.

**Example 16.** White noise: what’s in a name?