Plan Limit Theory (Countable-state case primarily)

1. Recurrence and Transience
2. Invariant Distributions
3. Limit Theorems

Next Time: General-state analogues, Extended Example

Reading: G&S 6.4, 6.5, 6.6
Homework 4 on-line tomorrow (sorry)
Homework solutions on-line up to date

Let $n, m \geq 0$. For a transition probability matrix $P$, we have:

$$P^{n+m} = P^n \cdot P^m,$$

(that is, $P^{n+m}(s, s') = \sum_{u \in S} P^n(s, u)P^m(u, s')$). (1)

For general transition kernels:

$$P^{n+m}(s, A) = \int_S P^n(s, du)P^m(u, A).$$ (2)

Review Theorem 2. If $X$ is Markov chain on countable state space $S$, then we can write

$$S = D \cup \bigcup_i C_i,$$ (3)

where the sets are disjoint and each $C_i$ is an absorbing, communicating class for the chain $X$.

Review Proposition 3. If $C \subset S$ is an absorbing, communicating class for a Markov chain $X$, then there exists an irreducible Markov Chain $X^C$ with state-space $C$ and whose transition probability kernel is given by $P_C(x, A) \equiv P(x, A \cap C)$ for $x \in C$.

Review Definition 4. For any state $s \in S$, define the period of $s$ by

$$d(s) = \gcd\{n \geq 1 \text{ such that } P^n(s, s) > 0\}.$$ (4)

This implies that $P^n(s, s) = 0$ unless $n = md(s)$ for some $m \in \mathbb{Z}_+$. An irreducible Markov Chain is said to be aperiodic if $d \equiv 1$.

Review Theorem 5. Let $X$ be an irreducible, countable-state Markov chain with common period $d$. Then, there are disjoint sets $U_1, \ldots, U_d \subset S$ such that

$$S = \bigcup_{k=1}^d U_k,$$ (5)

and

$$P(x, U_{k+1}) = 1 \quad \text{for } x \in U_k, \quad k = 0, \ldots, d - 1 \pmod{d}.$$ (6)

The sets $U_1, \ldots, U_d$ are called cyclic classes of $X$ because $X$ cycles through them successively.
**Useful Random Variables 6.** Let $X$ be a general-state Markov chain with state space $\mathcal{S}$. Let $A \subset \mathcal{S}$. Define

$$T_A = \inf \{ n \geq 1 \text{ such that } X_n \in A \} \quad (7)$$
$$S_A = \inf \{ n \geq 0 \text{ such that } X_n \in A \} \quad (8)$$
$$O_A = \sum_{n=1}^{\infty} 1\{X_n \in A\}. \quad (9)$$

These are called, respectively, the first return time, the first hitting time, and the occupation time of $A$. If $X_n$ never returns to or hits $A$, we take $T_A = \infty$ and $S_A = \infty$ respectively. As we will see, these random variables provide a great deal of information about the behavior of the chain.

For $A, B \subset \mathcal{S}$ and $s \in \mathcal{S}$, define

$$R(s, A) = P_s\{T_A < \infty\} \quad (10)$$
$$M(s, A) = E_s T_A \quad (11)$$
$$H(s, A) = P_s\{S_A < \infty\} \quad (12)$$
$$O(s, A) = E_s O_A \quad (13)$$
$$P^n_A(s, B) = P_s\{X_n \in B, T_A \geq n\}. \quad (14)$$

These are the return time probabilities, hitting probabilities, expected occupation times for the set $A$ and taboo probabilities for the set $B$ avoiding $A$, when the chain starts in state $s$.

Note that

$$O(s, A) = \sum_{n=1}^{\infty} P^n(s, A) \quad (15)$$
$$R(s, A) = \sum_{n=1}^{\infty} P^n_A(s, A). \quad (16)$$

**Definition 7.** The random variables $T_A$ and $S_A$ in the last item are stopping times, meaning that $\{T = n\} \in \sigma(X_0, \ldots, X_n)$ for every $n$.

Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ be the history of the chain up to time $n$. For a stopping time $T$, we can define the information in the chain up to time $T$ – the history up to time $T$ – as a $\sigma$-field $\mathcal{F}_T$ defined as follows:

$$\mathcal{F}_T = \{ A \in \mathcal{F} \text{ such that } A \cap \{T = n\} \in \mathcal{F}_n, \text{ for all } n \in \mathbb{Z}_+ \}. \quad (17)$$

**Theorem 8.** *The Strong Markov Property*

For any (discrete-time) Markov chain $X$ and a bounded, measurable function $h$ on sample paths,

$$E(h(X_{T+1}, X_{T+2}, \ldots) \mid \mathcal{F}_T) 1\{T < \infty\} = E_{X_T}(h(X_1, X_2, \ldots) 1\{T < \infty\}), \quad (18)$$

where $E_{X_T}$ corresponds to a chain whose initial distribution on $\mathcal{S}$ is the distribution of $X_T$. This is called the *Strong Markov Property*. 

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**Definition 9.** A set $A \subseteq \mathcal{S}$ is called **uniformly transient** if there exists $M < \infty$ such that $O(s, A) \leq M$ for all $s \in A$. $A$ is called **transient** if $O(s, A) < \infty$ for all $s \in A$. $A$ is called **recurrent** if $O(s, A) = \infty$ for all $s \in A$.

In particular, for a state $s \in \mathcal{S}$ of a countable-state Markov chain, we say that $s$ is uniformly transient/transient/recurrent if $\{s\}$ is.

**Theorem 10.** For an irreducible, countable-state Markov chain $X$, either $O(s, s') < \infty$ for all $s, s' \in \mathcal{S}$, in which case we say that $X$ is recurrent, or $O(s, s') = \infty$ for all $s, s' \in \mathcal{S}$ in which case we say that $X$ is transient.

Proof. Then since for any $u, v$, $u \rightarrow s$ and $s' \rightarrow v$, we can find an $\ell, m$ such that $P^\ell(u, s) > 0$ and $P^m(s', v) > 0$. Hence,

$$\sum_n P^{\ell+n+m}(u, v) > P^\ell(u, s) \left[ \sum_n P^n(s, s') \right] P^m(s', v).$$

It follows that $O(s, s') = \sum_{n=1}^{\infty} P^n(s, s') = \infty$ implies $O(u, v) = \infty$ and $O(u, v) < \infty$ implies $O(s, s') < \infty$. But both pairs of states were arbitrary, so the theorem is proved.

**Theorem 11.** Suppose $X$ is a countable-state Markov chain on $\mathcal{S}$. For any $s \in \mathcal{S}$, $O(s, s) \equiv O(s, \{s\}) = \infty$ if and only if $R(s, s) \equiv R(s, \{s\}) = 1$.

Hence, if $X$ is irreducible, either $R(s, s') = 1$ for all $s, s' \in \mathcal{S}$ or $R(s, s) < 1$ for all $s \in \mathcal{S}$.

To prove this theorem, we’ll use the same trick we used earlier in considering the return times of random walks. Notice that for any $s \in \mathcal{S}$ and any $n \geq 1$,

$$P^n(s, s) = \sum_{k=1}^{n} P_s \left\{ T_s = k \right\} P^{n-k}(s, s) = \sum_{k=0}^{n} P_s \left\{ T_s = k \right\} P^{n-k}(s, s),$$

where the latter follows because $P_s \left\{ T_s = 0 \right\} = 0$.

Let $G_s(z) = \sum_n P^n(s, s)z^n$ and $R_s(z) = \sum_n P_s \left\{ T_s = n \right\} z^n$. Then, we get

$$G_s(z) = 1 + G_s(z)R_s(z) \quad \Longrightarrow \quad G_s(z) = \frac{1}{1 - R_s(z)}.$$

Because $R_s(1) = P_s \left\{ T_s < \infty \right\}$ (or letting $z \rightarrow 1$ to be careful about convergence), we get that

$$R(s, s) = 1 \iff O(s, s) = \infty.$$

By Theorem 10 and equation (22), we have either $R(s, s) < 1$ for all $s$ or $R(s, s) = 1$ for all $s$. If the latter is true and $R(s, s') < 1$, then by irreducibility, we have $O(s', s) > 0$ and thus, for some $n$, $P^n(s', s) > 0$. This implies $R(s', s') < 1$, and the result follows by contradiction.

**Examples 12.**
1. Random Walk
2. Bounded Random Walk
3. Binomial Runs
4. Renewal Process and Forward Recurrence Time Chain
Definition 13. For $s \in \mathcal{S}$, recall $M(s, A) = \mathbb{E}_s T_A$. For $A = \{s\}$, $M(s, A) \equiv M(s, s)$. These are the expected return times to the set $A$ and the state $s$.

Definition 14. Let $X$ be a countable-state Markov chain on $\mathcal{S}$. If $s \in \mathcal{S}$ is a recurrent state, we call it positive recurrent if $M(s, s) < \infty$ and null recurrent if $M(s, s) = \infty$.

Definition 15. Let $X$ be a countable-state Markov chain on $\mathcal{S}$ with transition probabilities $P$. A (σ-finite) measure $\pi$ is an invariant measure for the chain if $\pi(s) \geq 0$ and $\pi(s') = \sum_s \pi(s) P(s, s')$, (23)

or in matrix terms $\pi = \pi \cdot P$, (24)

where we think of $\pi$ as a “row vector”.

An invariant measure $\pi$ is an invariant or stationary distribution, if in addition it is a probability mass function on $\mathcal{S}$.

Motivation 16. The names “invariant” and “stationary” come from the above properties. The first stems from the fact that $\pi$ does not change – is invariant – under the transition mechanism of the chain. The second comes from the fact that if the chain is started with initial distribution $\pi$, then the distribution of $X_n$ is given by $\pi \cdot P^n = \pi$. That is, the distribution of $X_n$ does not change with $n$.

Definition 17. If $X$ is an irreducible, recurrent, countable-state Markov chain on $\mathcal{S}$ and there exists a stationary distribution on $\mathcal{S}$, then $X$ is said to be a positive recurrent. Otherwise, $X$ is said to be null recurrent. Then, every state is, respectively, positive or null recurrent.

Theorem 18. If $X$ is an irreducible, recurrent, countable-state Markov chain, then there exists an invariant measure $\rho$ that is positive and unique up to constant multiples. The chain is positive recurrent if $\sum_s \rho(s) < \infty$ and null recurrent if $\sum_s \rho(s) = \infty$.

Theorem 19. If an irreducible, countable-state Markov chain has a stationary distribution, then it is unique and $\pi_s = 1/M(s, s)$.

Remarks 20. 1. Note that if $X$ admits a stationary distribution but is transient then $P^n(s, s') \to 0$ as $n \to \infty$, so

$$\pi_{s'} = \sum_s \pi_s P(s, s') \leq \sum_{s \in S_0} \pi_s P(s, s') + \sum_{s \not\in S_0} \pi_s$$

$$\to \sum_{s \not\in S_0} \pi_s \to 0,$$

(25)

for any finite $S_0 \subset \mathcal{S}$ with the last limit being as $S_0 \uparrow \mathcal{S}$. This is a contradiction, so $X$ must be recurrent.

2. See the proofs in G&S section 6.4.

To translate, define $q(s, s') = \sum_n P^n_{s,s'}(s', s')$ for the taboo probabilities $P^n_{s,s'}$. Notice that $M(s, s) = \sum_s q(s, s')$. G&S use $\rho_s(s') \equiv q(s, s')$ and $f_{s,s'}(n) \equiv P_X \{ T_{s'} = n \}$.

Examples 21.

- Doubly Stochastic, Finite Random Walk
- Canonical Random Walk
- Embedded MC for G/M/1 Queue