36-309/749
Experimental Design for Behavioral and Social Sciences

Sep. 22, 2015

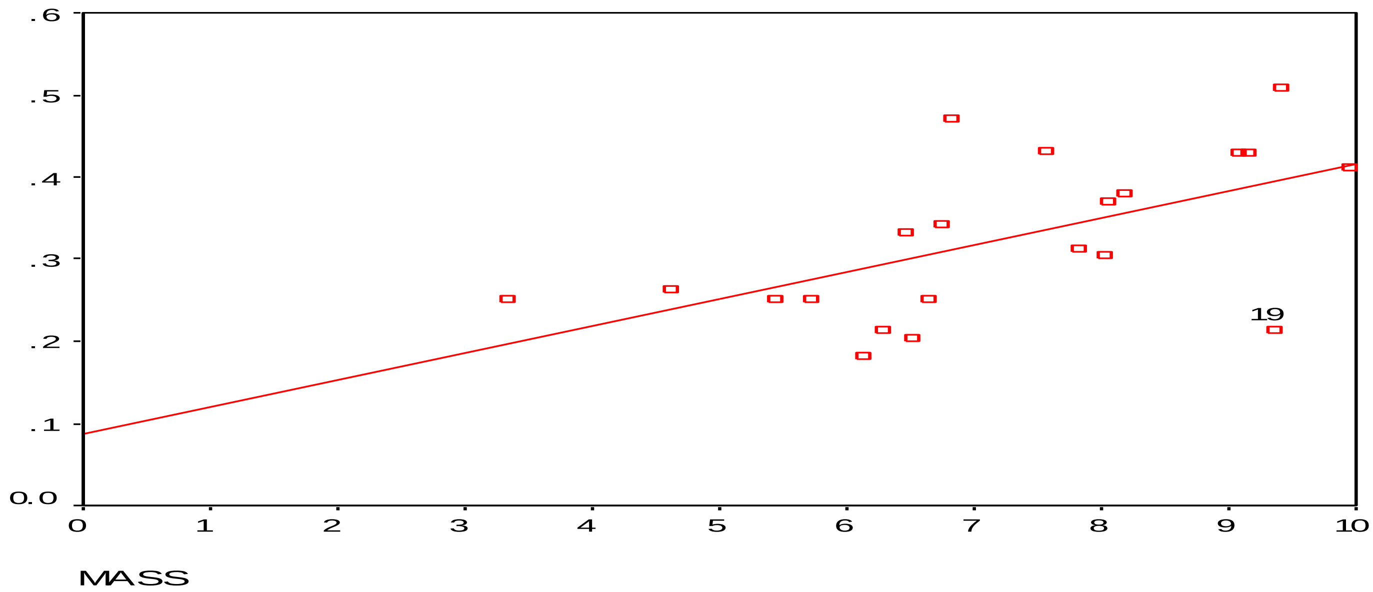
Lecture 4: Linear Regression

Simple Regression Example

Male black wheatear birds carry stones to the nest as a form of sexual display. Soler *et al.* wanted to find out whether there is a relationship between the mass (in g) of the stones carried and the immunologic health of the birds as measured by a test of T cell function (in mm).

Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
MASS	21	3.33	9.95	7.2043	1.70244
TCELL	21	.18	.51	.3240	.09674
Valid N (listwise)	21				



Example, cont.

Model		Unstandardized Coefficients		t	Sig.	95% Confidence Interval for B	
		B	Std. Error			Lower Bound	Upper Bound
1	(Constant)	.087	.079	1.112	.280	-.077	.252
	MASS	.033	.011	3.084	.006	.011	.055

ANOVA[b]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	.062	1	.062	9.513	.006[a]
	Residual	.125	19	.007		
	Total	.187	20			

a. Predictors: (Constant), MASS

b. Dependent Variable: TCELL

Regression: Main Ideas

- **Setting**: Quantitative outcome with a quantitative explanatory variable

- **Model**: (§9.1)
  - Structural (means) model: $E(Y|x) = \beta_0 + \beta_1 x$ (parameters are called "betas" or "coefficients"**)**
    - This defines a linear relationship, on average (linearity assumption).
    - The intercept, β_0, is the (population) mean of Y when x = 0.
    - The slope, β_1, is the (population) ***mean change*** in Y when x ***increases by 1***.

Model, cont.

- Fixed-x assumption: x's are measured with no (little) error
- Error model
  - "Errors" (deviations of Y from $\beta_0+\beta_1 x$) are **Gaussian** …
  - … with ***constant*** variance, σ^2, called "error variance"
  - … and the errors are **independent** of each other.

- Alternate model form: $Y_i|x_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad \varepsilon \sim N(0,\sigma^2)$ iid

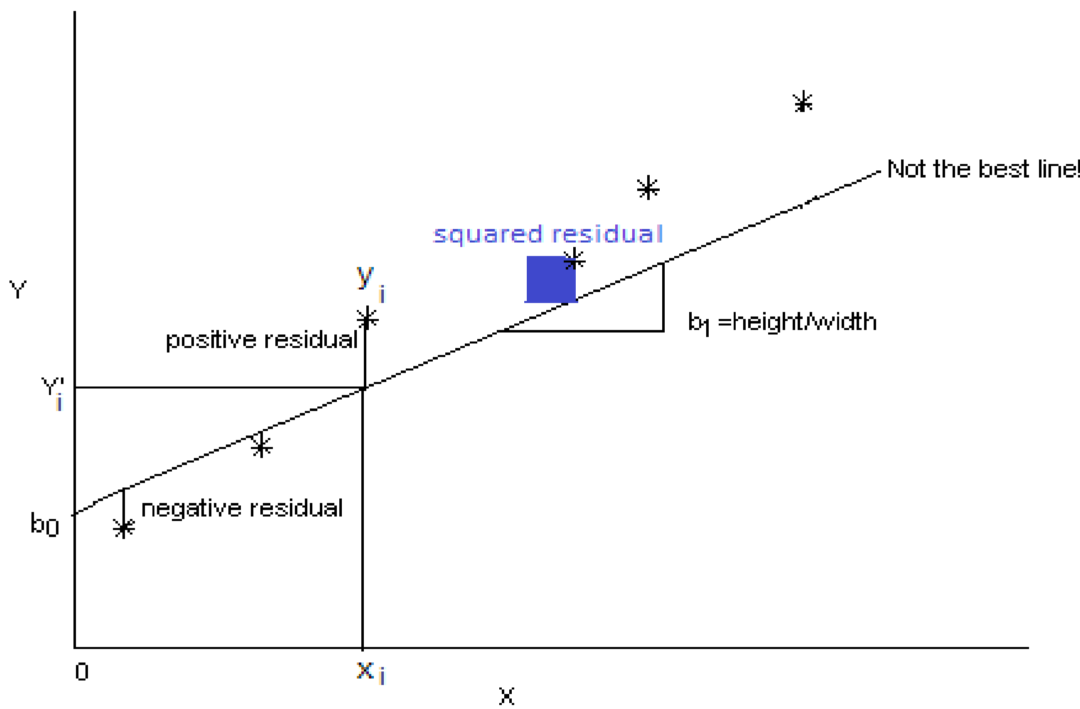
Main Ideas, cont.

- **Simple** vs. **multiple** regression

- **Null hypothesis:** H_0: $\beta_1=0$ vs. H_1: $\beta_1 \neq 0$.
  (Sometimes: H_0: $\beta_0=0$ vs. H_1: $\beta_0 \neq 0$
  or H_0: $\beta_1=1$ vs. H_1: $\beta_1 \neq 1$) (§9.2)
- **Interpolation:** good / **extrapolation:** bad

Main Ideas, cont.

- The "**least squares principle**" finds the best-fit line by minimizing the sum of squared residuals. (§9.4)



Main Ideas, cont.

- **Estimates:** "best" estimates of β_0 and β_1 are called b_0 and b_1 or $\widehat{\beta_0}$ and $\widehat{\beta_1}$ (read "**beta 0 hat**" and "**beta 1 hat**"). (§9.4, 9.5)

  - Technical & optional: The estimates of the coefficients are statistics that are unbiased, minimum variance estimates. For linear regression, "least squares" is the same as "maximum likelihood".

Main Ideas, cont.

- **Fitted (predicted) values**: $= \hat{Y}_i = Y_i' = \text{fit}_i = \text{pred}_i = \widehat{\beta_0} + \widehat{\beta_1} x_i = b_0 + b_1 x_i$. (§9.4)

- **Residual**: $\text{res}_i = r_i = Y_i - \text{fit}_i = Y_i - \hat{Y}_i$
  This is an estimate of the true "error". (§9.4, 9.6)

Main Ideas, cont.

- Optional estimation formulas:

$$b_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},\ b_0 = \bar{y} - b_1\bar{x},\ s^2 = \frac{\sum_{i=1}^{n} {r_i}^2}{n-2}$$

- **$\mathbf{Ms_{within}}$ = $\mathbf{Ms_{resid}}$ = $\mathbf{Ms_{error}}$ = $\mathbf{SS_{resid}/df}$** are exactly the same estimate of error variance, σ^2. (§9.8)
  - In general, the df equals (N - #betas).

Main ideas, cont.

- The Normality assumption leads to a Normal **sampling distribution of $\widehat{\boldsymbol{\beta}_1}$** with mean β_1 and a variance that is a complicated combination of σ^2 and the x values. When we substitute in the estimate of σ^2 and take the square root, we get a standard error (SE) of $\widehat{\beta_1}$ that can be used in a t-test (§9.4) of H_0: β_1=0:

- 95% **confidence interval** on β_1:
  $[b_1 \ - m\ \mathrm{SE}(b_1), b_1 + m\ \mathrm{SE}(b_1)]$, where m is approximately 2 and the exact value comes from finding the value of the appropriate t distribution that holds 95% of the distribution. (§9.4)

Main ideas, cont.

- **Goodness of fit /model comparison**: (§9.8)
  - R^2 (0 to 1) measures the fraction of the variability in the outcome "accounted for" by the explanatory variable(s). It is also r^2 in simple (one x) regression where r is the correlation of x and Y. (See slide 23 below for the formula.)
  - Adjusted R^2 corrects for the "more IVs ***always*** gives a higher R^2" problem.
  - Many consider AIC / BIC, which are "penalized likelihoods", even better for comparing models, e.g., deciding if some additional "x" should be included.

Main Ideas, cont.

- **Robustness** of the t-test in regression (§9.7)
  - moderately severe non-normality is OK (CLT)
  - mild to moderate unequal spread are OK (ratio<2)
  - only minimal correlation of errors is OK
  - non-fixed X is OK only if its variability is much less that the variability of Y
  - non-linearity is bad

SPSS Output and Interpretation

- SPSS has some mechanisms for trying different sets of explanatory variables (**model selection**) in multiple (>1 x) regression. The default is to include all specified x variables.

Variables Entered/Removed[b]

Model	Variables Entered	Variables Removed	Method
1	MASS[a]	.	Enter

a. All requested variables entered.

b. Dependent Variable: TCELL

SPSS Output and Interp., cont.

- There is a statistically significant positive **association** between the mass of stones collected and the T cell activity (reject H_0: $\beta_1=0$ or equivalently $\beta_{MASS}=0$; $p=0.006$). ***If*** the levels of the explanatory variable were randomly assigned, we could say that a ***rise*** in stone mass **<u>causes</u>** a ***rise*** in T cell function.
  - Concluding only that "stone mass is statistically significant" is stupid!!!

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	t	Sig.	95% Confidence Interval for B Lower Bound	95% Confidence Interval for B Upper Bound
1	(Constant)	.087	.079	1.112	.280	-.077	.252
	MASS	.033	.011	3.084	.006	.011	.055

a. Dependent Variable: TCELL

SPSS Output and Interp., cont.

- The three major **causes of an association between variables** X and Y are that changes in X cause changes in Y, changes in Y cause changes in X, and that changes in a third variable causes changes in both X and Y. In addition, type 1 error can result in an ***apparent*** association between X and Y when there really is none (a type-1 error).

- **Interpretation of slope coefficient**: A rise of 1 gram of mass is associated with an estimated mean rise of 0.033 mm in T cell function. It is also OK (better) to say that, e.g., a rise of 10 gram of mass is associated with an estimated 0.33 mm mean rise in T cell function.

SPSS Output and Interp., cont.

- The intercept is ***the mean of Y when x is 0.***
  - In this example, since x=0 is far from the rest of the data, an **interpretation of the intercept** would be an *unwarranted extrapolation*. If we were to make that extrapolation we would express it as “the *predicted* (estimated) *mean* of T cell function for wheatears that carry *no* stones is 0.087 mm”. Here, the non-significant p-value (and the CI) tell us that we cannot rule out that the mean T cell response for 0 g of stones is 0 mm (cannot reject H_0:β_0=0).
  - ***Only interpret the intercept p-value when x=0 makes sense <u>and</u> when we have data at or near x=0 <u>and</u> when we care if E(Y|x=0) equals zero or not, i.e. was originally unknown.***

SPSS Output and Interp., cont.

- The "**CI** for B_{MASS}" has the **interpretation** that "we are *95% confident* that the *mean rise* in T cell function associated with a *one* gm *increase* in stone mass is between 0.011 and 0.055."
  - Technically, when the assumptions of the model are met, CI's constructed by the standard recipe used here will include the true coefficient value 95% of the time (over repeated experiments, whether or not the null hypothesis is true, but only when the assumptions are true).

SPSS Output and Interp., cont.

- The **mean squared error (MSE)** is 0.007 which is the best estimate of σ^2. Our model predicts that for any given x value, the distribution of the outcome for many subjects with that same x value will be normally distributed with mean $\beta_0+\beta_1 x$ and variance 0.007 and sd=$\sqrt{0.007}$=0.081 and 2sd=0.162.

ANOVA[b]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	.062	1	.062	9.513	.006[a]
	Residual	.125	19	.007		
	Total	.187	20			

a. Predictors: (Constant), MASS

b. Dependent Variable: TCELL

SPSS Output and Interp., cont.

- The "case-wise diagnostics" tries to detect possible **outliers**. Here it tells us that the subject farthest (vertically) from the best fit line is subject 19, who has actual outcome 0.21 mm, predicted outcome 0.39 mm, and residual –0.18. So this bird had a T cell measurement 0.18 lower than "expected" by the model. The **standardized residual** divides by the standard deviation of the residuals; –2.24 is not highly unlikely.

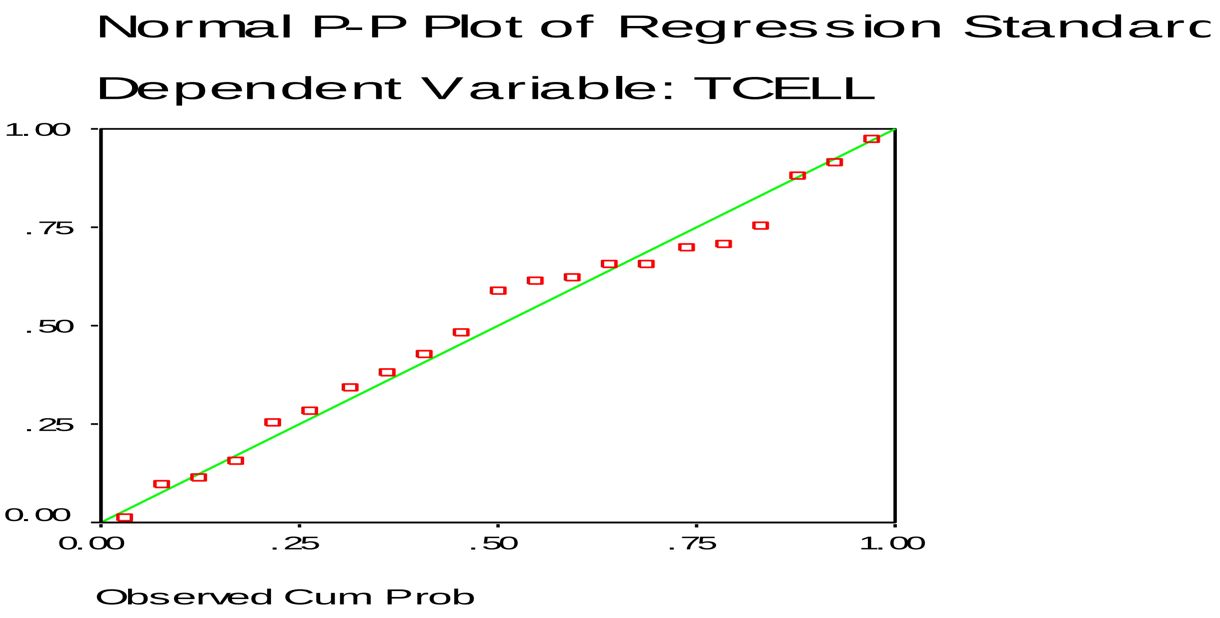
Casewise Diagnostics[a]

Case Number	Std. Residual	TCELL	Predicted Value	Residual
19	-2.239	.21	.3944	-.1814

a. Dependent Variable: TCELL

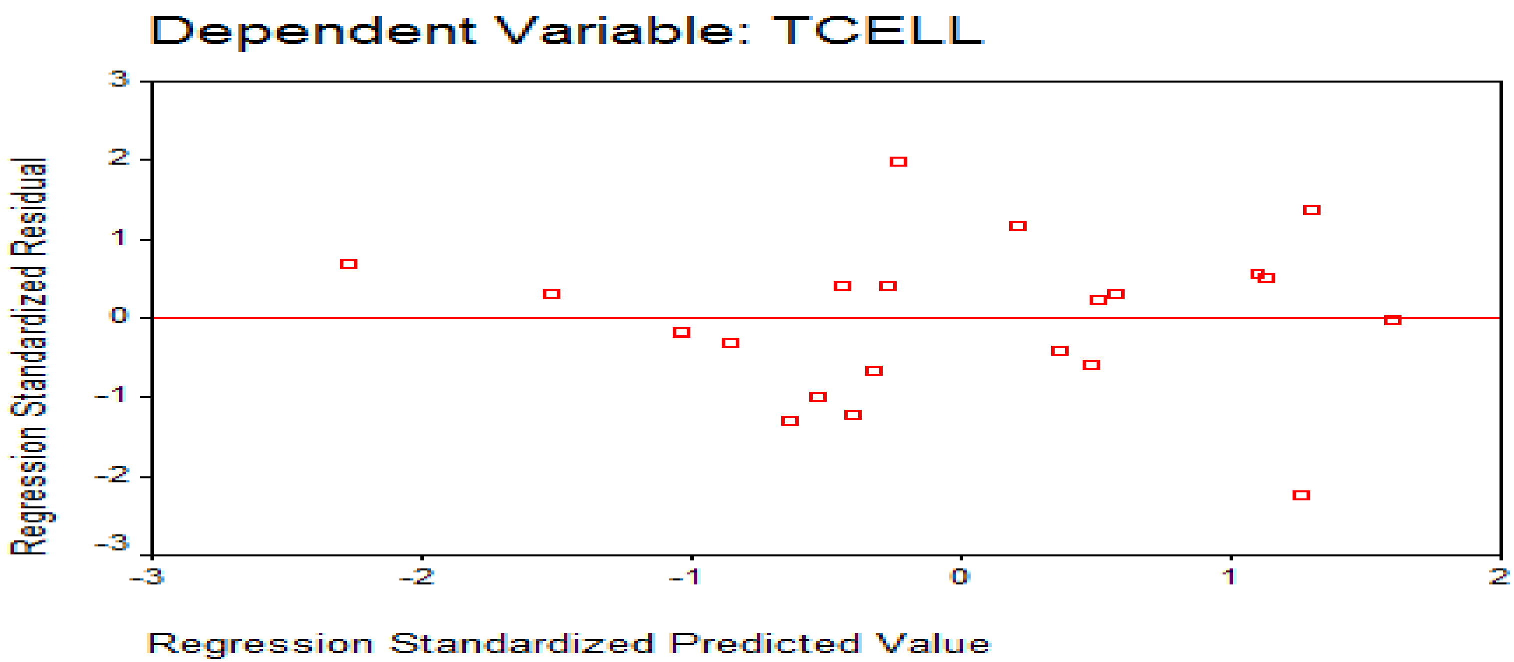
SPSS Output and Interp., cont.

- The "Normal P-P" (similar to the quantile-quantile or **quantile-normal**) **plot** of the residuals helps test the assumption of a normal distribution of the Y values at each x by combining the residuals from all x values.



SPSS Output and Interp., cont.

- The **residual vs. fit plot** of "Standardized Residual" vs "Standardized Predicted Value" helps ***test the assumptions of linearity and equal spread***. (Alternatively un-standardized values are used, but they are a bit harder to get in SPSS.) Use the "vertical band" method of interpretation.



SPSS Output and Interp., cont.

- The "**R-Squared**" value of 0.334 indicates that 33.4% of the overall variation in the outcome has been "explained" by the explanatory variable(s).
  - $SS_{Total} = \Sigma(y_i-\bar{y})^2$, $SS_{Error} = \Sigma(y_i-\hat{y})^2$, $SS_{explained} = SS_{Total} - SS_{Error}$
  - $R^2 = (SS_{Total}-SS_{Error}) / SS_{T:}$ fraction of the total deviations that is explained by the explanatory variable(s)
  - R^2 values range from 0 to 1 (high in physics, low in psychology)
  - ***Adjusted***-R^2 corrects for "more is always better" and is more realistic
  - AIC/BIC are even better for model comparison/selection

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.578[a]	.334	.299	.08102

a. Predictors: (Constant), MASS

b. Dependent Variable: TCELL

Conclusion

There are lots of useful regression results beyond just a p-value!