Confidence Ellipsoids for Bivariate Normals

Consider random variables $[X_1\ X_2]^T \sim N([\mu_1\ \mu_2]^T, \Sigma)$ where $\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$.

$$
\begin{aligned}
f(X) &= (2\pi|\Sigma|)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}[X_1-\mu_1\ X_2-\mu_2]\ \Sigma^{-1} \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}\right) \\
\Sigma^{-1} &= \begin{bmatrix} \sigma_2^2/|\Sigma| & -\sigma_{12}/|\Sigma| \\ -\sigma_{12}/|\Sigma| & \sigma_1^2/|\Sigma| \end{bmatrix} \\
|\Sigma| &= \sigma_1^2\sigma^2 - \sigma_{12}^2 \\
f(X) &= (2\pi|\Sigma|)^{-\frac{1}{2}} \exp\left(\frac{-1}{2|\Sigma|}\left[\ \sigma_2^2(X_1-\mu_1) - \sigma_{12}(X_2-\mu_2),\ \sigma_1^2(X_2-\mu_2) - \sigma_{12}(X_1-\mu_1)\ \right]\begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}\right) \\
&= (2\pi|\Sigma|)^{-\frac{1}{2}} \exp\left(\frac{-1}{2|\Sigma|}\left[\sigma_2^2(X_1-\mu_1)^2 + \sigma_1^2(X_2-\mu_2)^2 - 2\sigma_{12}(X_1-\mu_1)(X_2-\mu_2)\right]\right)
\end{aligned}
$$

For a confidence ellipsoid defined by k, X_2 coordinates at $X_1 = x_1$ are solutions of

$$k = \sigma_1^2 X_2^2 + (2\sigma_{12}\mu_1 - 2\sigma_1^2\mu_2 - 2\sigma_{12}x_1)X_2 + \left[\sigma_2^2 x_1^2 - 2\sigma_2^2\mu_1 x_1 + 2\sigma_{12}\mu_2 x_1\right]$$

X_2 coordinates for "tips" of the ellipsoid for constant k are

$$X_2 = \frac{-(2\sigma_{12}\mu_1 - 2\sigma_1^2\mu_2 - 2\sigma_{12}\hat{x}_1)}{2\sigma_1^2} = \mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(\hat{x}_1 - \mu_1)$$

where $\hat{x}_1$ is the solution to:

$$
\begin{aligned}
0 &= (2\sigma_{12}\mu_1 - 2\sigma_1^2\mu_2 - 2\sigma_{12}x_1)^2 - 4\sigma_1^2\left[\sigma_2^2 x_1^2 - 2\sigma_2^2\mu_1 x_1 + 2\sigma_{12}\mu_2 x_1 - k\right] \\
0 &= (4\sigma_{12}^2 - 4\sigma_1^2\sigma_2^2)x_1^2 + (8\sigma_1^2\sigma_{12}\mu_2 - 8\sigma_{12}^2\mu_1 + 8\sigma_1^2\sigma_2^2\mu_1 - 8\sigma_1^2\sigma_{12}\mu_2)x_1 \\
&+ (4\sigma_{12}^2\mu_1^2 + 4\sigma_1^4\mu_2^2 - 8\sigma_{12}\sigma_1^2\mu_1\mu_2 + 4\sigma_1^2 k)
\end{aligned}
$$

So for a given k,

$$\hat{x}_1 = \frac{-(8\sigma_1^2\sigma_{12}\mu_2 - 8\sigma_{12}^2\mu_1 + 8\sigma_1^2\sigma_2^2\mu_1 - 8\sigma_1^2\sigma_{12}\mu_2) \pm \sqrt{q}}{8\sigma_{12}^2 - 8\sigma_1^2\sigma_2^2}$$

where $q = (8\sigma_1^2\sigma_{12}\mu_2 - 8\sigma_{12}^2\mu_1 + 8\sigma_1^2\sigma_2^2\mu_1 - 8\sigma_1^2\sigma_{12}\mu_2)^2 - 4(4\sigma_{12}^2 - 4\sigma_1^2\sigma_2^2)(4\sigma_{12}^2\mu_1^2 + 4\sigma_1^4\mu_2^2 - 8\sigma_{12}\sigma_1^2\mu_1\mu_2 + 4\sigma_1^2 k)$

For any given k, the area inside the ellipsoid can be calculated by constructing a grid between the two values of $\hat{x}_1$ and then summing the density over a grid over the range of X_2.