Approximations for Mean and Variance of a Ratio

Consider random variables $R$ and $S$ where $S$ either has no mass at 0 (discrete) or has support $[0, \infty)$. Let $G = g(R, S) = R/S$. Find approximations for $EG$ and $\text{Var}(G)$ using Taylor expansions of $g()$.

For any $f(x, y)$, the bivariate first order Taylor expansion about any $\theta = (\theta_x, \theta_y)$ is

$$f(x, y) = f(\theta) + f'_x(\theta)(x - \theta_x) + f'_y(\theta)(y - \theta_y) + R$$

(1)

where $R$ is a remainder of smaller order than the terms in the equation.

Switching to random variables with finite means $EX \equiv \mu_x$ and $EY \equiv \mu_y$, we can choose the expansion point to be $\theta = (\mu_x, \mu_y)$. In that case the first order Taylor series approximation for $f(X, Y)$ is

$$f(X, Y) = f(\theta) + f'_x(\theta)(X - \mu_x) + f'_y(\theta)(Y - \mu_y) + R$$

(2)

The approximation for $E(f(X, Y))$ is therefore

$$E(f(X, Y)) = E[f(\theta)] + E[f'_x(\theta)](X - \mu_x) + E[f'_y(\theta)](Y - \mu_y) + R$$

(3)

$$= E[f(\theta)] + E[f'_x(\theta)]EX + E[f'_y(\theta)]EY + R$$

(4)

$$= E[f(\theta)] + 0 + 0$$

(5)

$$= f(\mu_x, \mu_y)$$

(6)

Note that if $f(X, Y)$ is a linear combination of $X$ and $Y$, this result matches the well-known result from mathematical statistics that $E(aX + bY) = aEX + bEY = a\mu_x + b\mu_y$, and in that case the error of approximation is zero. But with the Taylor series expansion, we have extended that result to non-linear functions of $X$ and $Y$.

For our example where $f(x, y) = x/y$ the approximation is $E(X/Y) = E(f(X, Y)) = f(\mu_x, \mu_y) = \mu_x/\mu_y$.

The second order Taylor expansion is

$$f(x, y) = f(\theta) + f'_x(\theta)(x - \theta_x) + f'_y(\theta)(y - \theta_y)$$

$$+ \frac{1}{2} \left\{ f''_{xx}(\theta)(x - \theta_x)^2 + 2f''_{xy}(\theta)(x - \theta_x)(y - \theta_y) + f''_{yy}(y - \theta_y)^2 \right\} + R$$

(8)

So a better approximation is for $E[f(X, Y)]$ expanded around $\theta = (\mu_x, \mu_y)$ is

$$E(f(X, Y)) \approx f(\theta) + \frac{1}{2} \left\{ f''_{xx}(\theta)\text{Var}(X) + 2f''_{xy}(\theta)\text{Cov}(X, Y) + f''_{yy}(\theta)\text{Var}(Y) \right\}.$$  

(9)

Note that we again use the fact that $E(X - \mu_x) = 0$, and we now add in the definitions for variance and covariance: $\text{Var}(X) = E[(X - \mu_x)^2]$ and $\text{Cov}(X) = E[(X - \mu_x)(Y - \mu_y)]$. 
For \( f(R, S) = R/S \), the derivatives are \( f''_{RR}(R, S) = 0 \), \( f''_{RS}(R, S) = -S^{-2} \), and \( f''_{SS}(R, S) = \frac{2R}{S^3} \).

Specifically, when \( \theta = (\mu_R, \mu_S) \), we have \( f(\theta) = \mu_R/\mu_S \), \( f''_{RR}(\theta) = 0 \), \( f''_{RS}(\theta) = -\frac{1}{(\mu_S)^2} \), and \( f''_{SS}(\theta) = \frac{2\mu_R}{(\mu_S)^3} \).

Then an improved approximation of \( E(R/S) \) is

\[
E(R/S) \equiv E(f(R, S)) \approx \frac{\mu_R}{\mu_S} - \frac{\text{Cov}(R, S)}{(\mu_S)^2} + \frac{\text{Var}(S)\mu_R}{(\mu_S)^3}
\]

(11)

By the definition of variance, the variance of \( f(X, Y) \) is

\[
\text{Var}(f(X, Y)) = E \left\{ [f(X, Y) - E(f(X, Y))]^2 \right\}
\]

(12)

Using \( E(f(X, Y)) \approx f(\theta) \) (from above)

\[
\text{Var}(f(X, Y)) \approx E \left\{ [f(X, Y) - f(\theta)]^2 \right\}
\]

(13)

Then using the first order Taylor expansion for \( f(X, Y) \) expanded around \( \theta \)

\[
\text{Var}(f(X, Y)) \approx E \left\{ \left[ f'_{x}(\theta)(X - \theta_x) + f'_{y}(\theta)(Y - \theta_y) \right]^2 \right\}
\]

(14)

\[
= E \left\{ \left[ f'_{x}(\theta)(X - \theta_x) + f'_{y}(\theta)(Y - \theta_y) \right]^2 \right\}
\]

(15)

\[
= E \left\{ f'^2_{x}(\theta)(X - \theta_x)^2 + 2f'_{x}(\theta)(X - \theta_x)f'_{y}(\theta)(Y - \theta_y) + f'^2_{y}(\theta)(Y - \theta_y)^2 \right\}
\]

(16)

\[
= f'^2_{x}(\theta)\text{Var}(X) + 2f'_{x}(\theta)f'_{y}(\theta)\text{Cov}(X, Y) + f'^2_{y}(\theta)\text{Var}(Y)
\]

(17)

Now we return to our example: \( f(R, S) = R/S \) expanded around \( \theta = (\mu_R, \mu_S) \).

Since \( f'_R = S^{-1}, f'_S = -\frac{R}{S^2} \) and \( \theta = (\mu_R, \mu_S) \), we now have \( f'^2_R(\theta) = \frac{1}{(\mu_S)^2}, \ f'_R(\theta)f'_S(\theta) = -\frac{\mu_R}{(\mu_S)^3}, \ f''_{SS}(\theta) = \frac{(\mu_R)^2}{(\mu_S)^3} \), and so

\[
\text{Var}(R/S) \approx \frac{1}{(\mu_S)^2} \text{Var}(R) + \frac{2}{(\mu_S)^2} \frac{\mu_R}{\mu_S} \text{Cov}(R, S) + \frac{(\mu_R)^2}{(\mu_S)^4} \text{Var}(S)
\]

(18)

\[
= \frac{(\mu_R)^2}{(\mu_S)^2} \left[ \frac{\text{Var}(R)}{(\mu_R)^2} - \frac{2\text{Cov}(R, S)}{\mu_R \mu_S} + \frac{\text{Var}(S)}{(\mu_S)^2} \right]
\]

(19)

\[
= \frac{\sigma^2_R}{(\mu_R)^2} - \frac{2\text{Cov}(R, S)}{\mu_R \mu_S} + \frac{\sigma^2_S}{(\mu_S)^2}
\]

(20)
