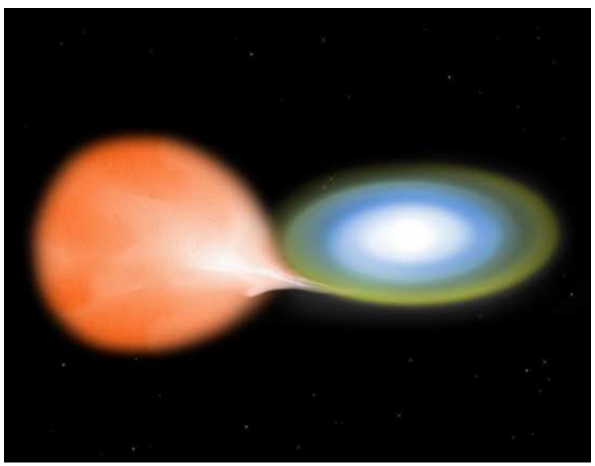
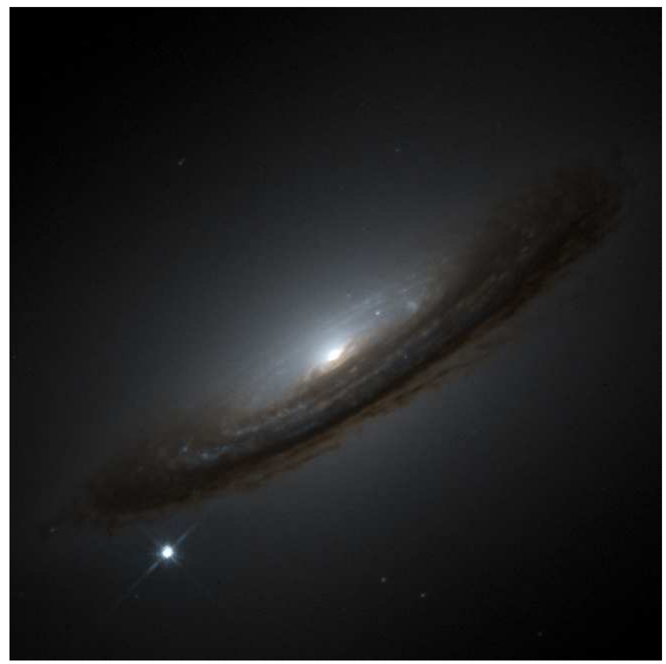
Non-Parametric Analysis of Supernova Data and the Dark Energy Equation of State

P. E. Freeman, C. R. Genovese, & L. Wasserman

Department of Statistics · Carnegie Mellon University

E-mail: pfreeman@cmu.edu





INTRODUCTION

$\Rightarrow$ Analyses of observed Type Ia supernovae (SNe) suggest that the universe is accelerating, not decelerating as once assumed.

$\Rightarrow$ One possibility is that "dark energy," characterized by negative pressure, drives the acceleration.

$\Rightarrow$ The dark energy equation of state (EOS) parameter is the ratio of dark energy pressure to dark energy density:

$$w(z) \;=\; P_{\mathrm{DE}}(z)/\rho_{\mathrm{DE}}(z)$$

where z denotes the redshift (a proxy for lookback time).

$\Rightarrow$ The determination of $w(z)$ allows us to differentiate between competing theories of dark energy.

$\rightarrow$ *e.g.*, $w(z) = -1$ is Einstein's Cosmological Constant model, his self-described "greatest blunder."

THE DATA

$\Rightarrow$ "Gold" SNe dataset of Riess *et al.* (2004): 156 SNe.

$\Rightarrow$ Data: redshifts z *vs.* distance moduli μ (with errors σ_μ).

$\Rightarrow$ Transform μ to log of the co-moving distance r_l:

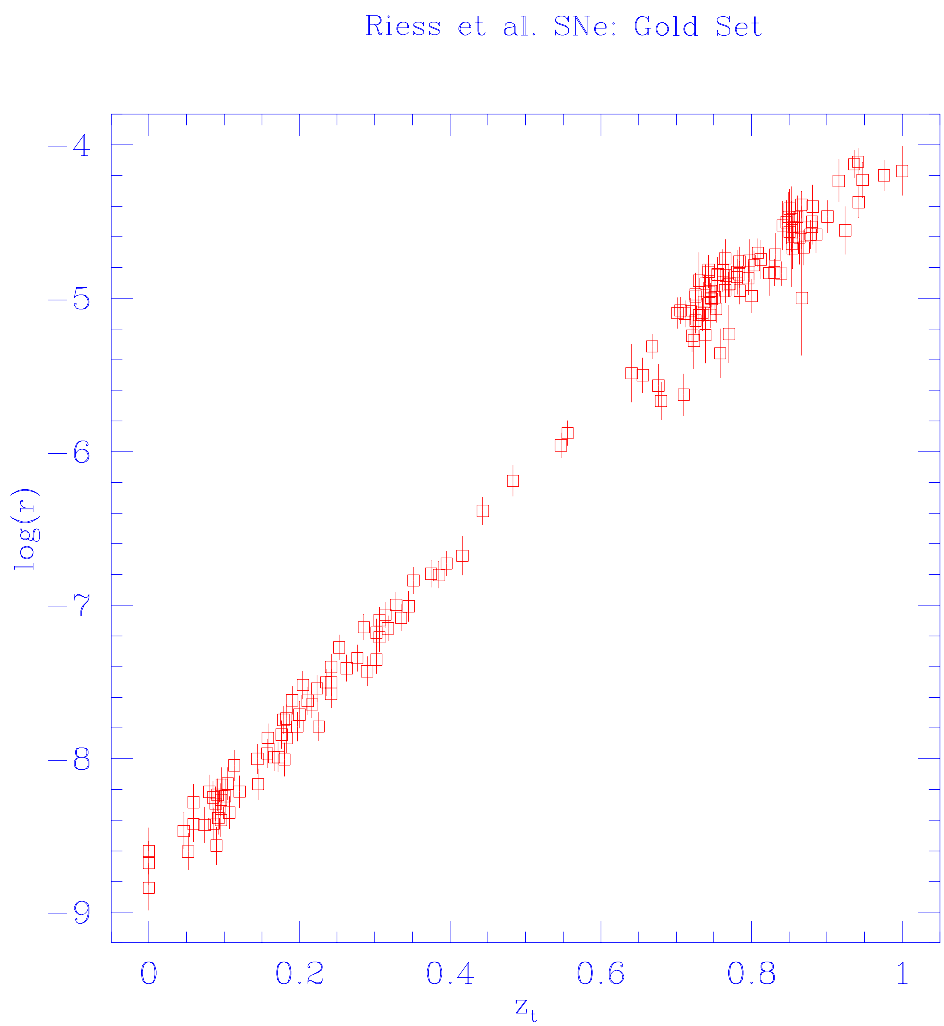
$$r_l(z) = \log\ r(z) = \log\left[\frac{1}{c(1+z)}10^{\frac{\mu-25}{5}}\right] = \log(10)\left(\frac{\mu}{5}-5\right) - \log(1+z) - \log c$$

$$\mathbb{V}_{r_l}(z) = \left(\frac{\partial r_l}{\partial \mu}\right)^2 \sigma_\mu^2 + \left(\frac{\partial r_l}{\partial z}\right)^2 \sigma_z^2 \approx \left(\frac{\log(10)\sigma_\mu}{5}\right)^2 .$$

c is the speed of light and the unit of distance is Mpc/(km/s).

$\Rightarrow$ Perform log transform $z \rightarrow z_t \in [0,1]$:

$$z_t = 1 - \frac{\log z'}{\min(\log z')}, \quad \text{where} \quad z' = \frac{z}{\max(z)} .$$


Riess et al. SNe: Gold Set
log(r)
−4
−5
−6
−7
−8
−9
0
0.2
0.4
0.6
0.8
1
z_t

DATA ANALYSIS

$\Rightarrow$ In this work, we estimate:

- $\rightarrow$ k^{th} derivatives $r_l^{(k)}(z_t)$ using local polynomial regression;
- $\rightarrow$ optimal bandwidths $h^{(k)}$ for each derivative function; and
- $\rightarrow$ proper confidence bands for each derivative function.

$\Rightarrow$ *cf.* Daly & Djorgovski (2003, 2004, 2005), who perform the first step only (while choosing bandwidths by eye and using conventional confidence bands output by the *Numerical Recipes* routine `FIT`).

COSMOLOGICAL PARAMETERS q AND w

⇒ $q(z)$ characterizes the *deceleration* of the universe:

$$q(z) = -\left(\frac{\ddot{a}a}{\dot{a}^2}\right) = -\left[1 + (1+z)\frac{r^{(2)}}{r^{(1)}}\right].$$

⇒ $w(z)$ characterizes the dark energy EOS:

$$\begin{aligned} P_{\rm DE}(z) &= -\rho_{\rm crit}\left(\frac{1}{H_{\rm o}r^{(1)}}\right)^2\left[1 + \frac{2}{3}(1+z)\frac{r^{(2)}}{r^{(1)}}\right] \\ \rho_{\rm DE}(z) &= \rho_{\rm crit}\left[\left(\frac{1}{r^{(1)}}\right)^2 - \Omega_{\rm M}(1+z)^3\right] \\ w(z) &= \frac{P_{\rm DE}(z)}{\rho_{\rm DE}(z)} = \frac{1+z}{3}\frac{3H_{\rm o}^2\Omega_{\rm M}(1+z)^2 + 2r^{(2)}/(r^{(1)})^3}{H_{\rm o}^2\Omega_{\rm M}(1+z)^3 - 1/(r^{(1)})^2} - 1\,. \end{aligned}$$

$\rho_{\rm crit}$ is the density of a flat universe, $\Omega_{\rm M}$ is the fractional contribution of matter to $\rho_{\rm crit}$, and $H_{\rm o}$ is the Hubble constant.

⇒ Assumptions necessary to express $w(z)$: the universe is homogeneous, isotropic, and spatially flat; its metric is the FRW metric; and gravity is described by general relativity.

LOCAL POLYNOMIAL REGRESSION

⇒ We fit a locally weighted p-degree polynomial at each of n data points $z_{t,i}$ to estimate the k^{th} derivative $r_l^{(k)}(z_t)$:

$$\hat{r}_l^{(k)}(z_t) \approx \frac{d^k}{dz_t^k} \sum_{i=0}^{p} a_i \frac{(z_{t,i} - z_t)^i}{i!} = a_k \, .$$

⇒ We minimize the weighted residual sum of squares:

$$\begin{aligned} RSS &= (r_l - \mathbf{Z}a)^T \mathbf{W} (r_l - \mathbf{Z}a), \quad \text{where} \\ Z_{i,j} &= \frac{(z_{t,i} - z_t)^j}{j!} \quad \text{and} \quad W_{i,j} = \begin{cases} \frac{1}{\sigma^2_{r_l,i}} \frac{1}{h} K\left(\frac{z_{t,i} - z_t}{h}\right) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} . \end{aligned}$$

K is the kernel function ($e.g.$, Gaussian).

⇒ If $\mathbf{WZ}$ has full column rank, then $\hat{a}$ is given exactly by

$$\hat{a} = (\mathbf{Z}^T \mathbf{W} \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{W} r_l \, .$$

OPTIMAL BANDWIDTH: MINIMIZING RISK

⇒ The optimal bandwidth h minimizes the risk function:

$$\mathbb{R}(r_l^{(k)}, \hat{r}_l^{(k)}) \;=\; \mathbb{B}^2(r_l^{(k)}, \hat{r}_l^{(k)}) + \mathbb{V}(\hat{r}_l^{(k)})\,.$$

⇒ $\mathbb{V}$ is the variance, which we may directly compute:

$$V(\hat{r}_l^{(k)}) \;=\; (\mathbf{Z}^T\mathbf{W}\mathbf{Z})^{-1}(\mathbf{Z}^T\mathbf{\Sigma}\mathbf{Z})(\mathbf{Z}^T\mathbf{W}\mathbf{Z})^{-1}, \text{ where } \Sigma_{i,j} \;=\; \begin{cases} \frac{1}{h^2}K^2(\frac{z_{t,i}-z_t}{h})\sigma_{r_l}^2 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

⇒ $\mathbb{B}$ is the bias, which is *not* directly computable.

⇒ A risk estimator for $r_l^{(0)}$ is generalized cross-validation (GCV):

$$\mathrm{GCV}(\hat{r}_l^{(0)}) \;=\; n\frac{\sum_{i=1}^{N}(r_l^{(0)} - \hat{r}_l^{(0)})^2}{(n-\nu)^2} \text{ where } \nu \;=\; \mathrm{Tr}(L) \text{ with } \hat{r}^{(0)} \;=\; \mathbf{L}r_l\,.$$

⇒ One *cannot* apply GCV to function derivatives $r_l^{(k>0)}$!

OPTIMAL BANDWIDTH: BIAS ESTIMATION

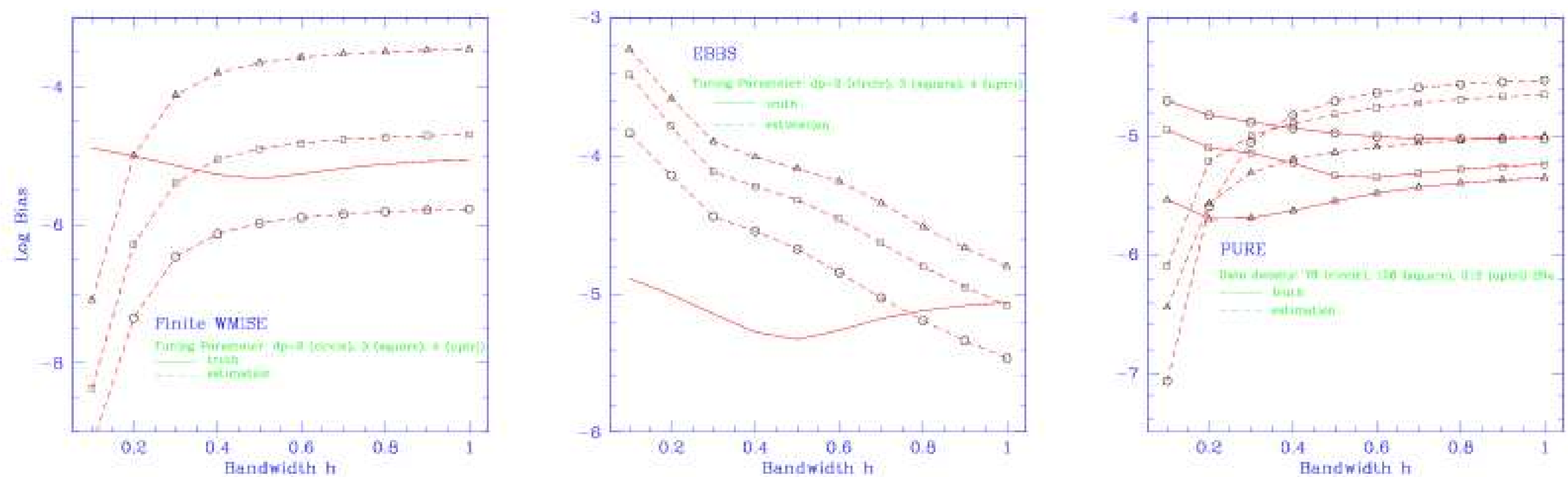
$\Rightarrow$ We have tested three bias estimators:

$\longrightarrow$ Finite Sample WMISE (§4.3, Fan & Gjibels 1996);

$\longrightarrow$ EBBS (Ruppert 1996); and

$\longrightarrow$ PURE (Lee & Solo 1999).

$\Rightarrow$ Typical results for $r_l^{(0)}$ (156 uniformly sampled SNe):



$\Rightarrow$ None of these estimators performs well with the current data!

CONFIDENCE BANDS: COMPUTATION

$\Rightarrow$ Conventional confidence band: for given z_t, $100(1-\alpha)$ percent chance that the truth lies within the confidence band.

$\Rightarrow$ Better choice: a confidence band defined such that there is a $100(1-\alpha)$ percent chance that all points of the (smoothed) true function lie within the band (see Sun & Loader 1994).

$\Rightarrow$ Solve the following for c:

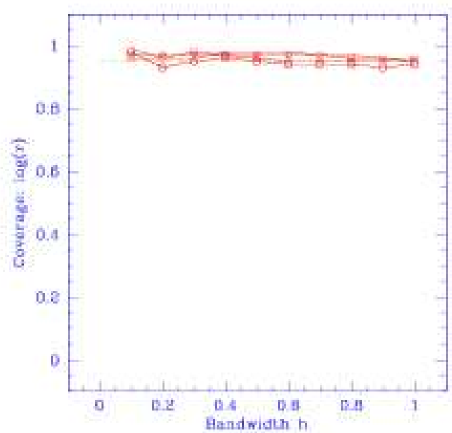
$$1-\alpha \;\approx\; 2(1-\Phi(c)) + \frac{\kappa_0}{\pi}e^{-c^2/2} \text{ where } \begin{aligned} \kappa_0 &= \int \| \frac{\partial T^{(k)}}{\partial z_t} \| \, dz_t \\ T_i^{(k)}(z_t) &= \frac{\sigma_{r_l,i} l_i^{(k)}(z_t)}{\|\sigma_{r_l} l^{(k)}(z_t)\|} \\ l^{(k)}(z_t)^T &= e_{k+1}^T(\mathbf{Z}^T\mathbf{W}\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{W} \end{aligned}$$

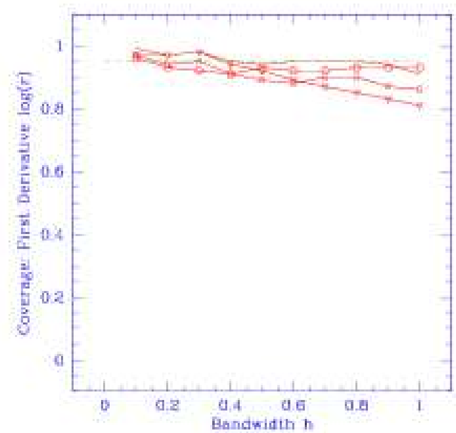
$\Rightarrow$ The confidence band is

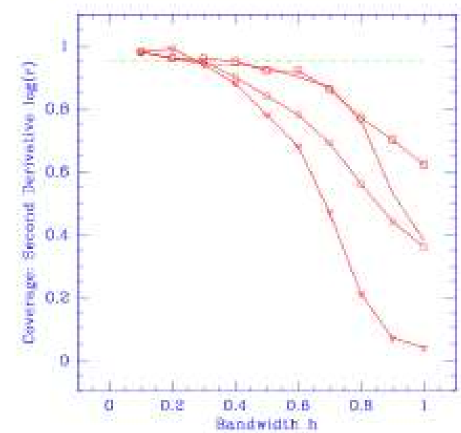
$$\hat{r}_l^{(k)}(z_t) \;\pm\; c\sqrt{V[\hat{r}_l^{(k)}(z_t)]} \text{ where } V[\hat{r}_l^{(k)}(z_t)] \;=\; \sum_{i=1}^{n}(\sigma_{r_l,i} l_i^{(k)}(z_t))^2$$

CONFIDENCE BANDS: PERFORMANCE

⇒ Performance is as expected for $r_l^{(0)}$, but the estimated bands become too small in the high-(h, k, n) limit.







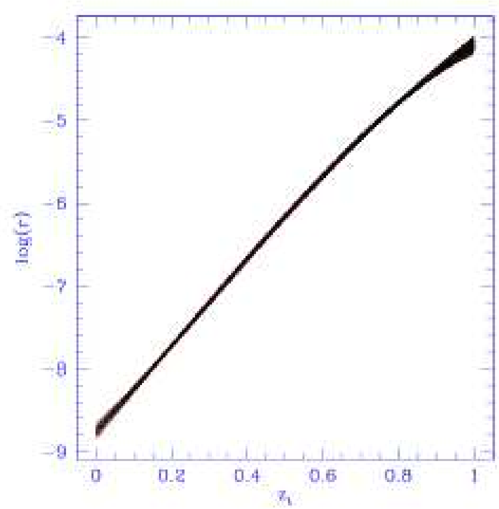
(PRELIMINARY) RESULT: $r_l^{(k)}$

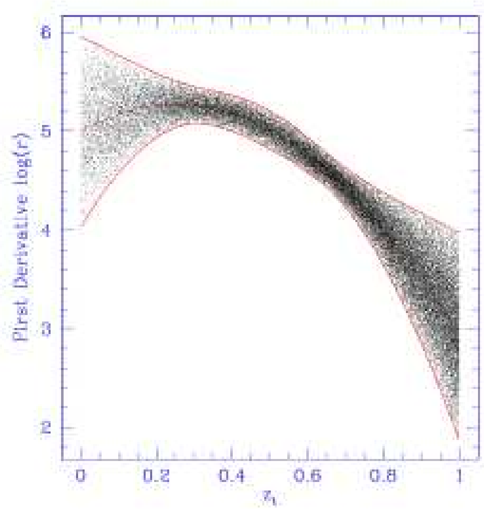
⇒ The use of GCV indicates $h^{(0)} \approx 1$. We assume $h^{(k)} = 1 \; \forall \; k$.

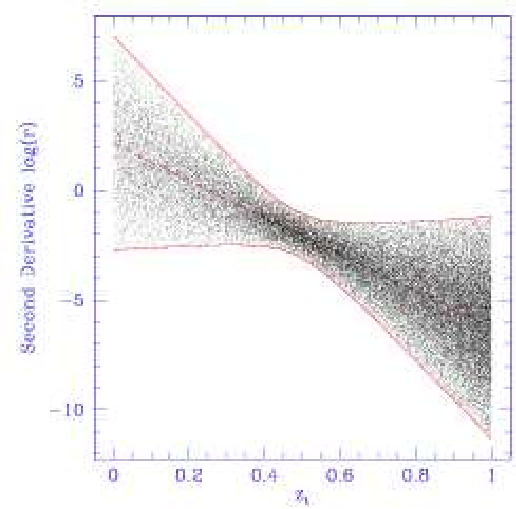
⇒ We set $p = 3$, *i.e.*, we fit with a local cubic polynomial.

⇒ We do not adjust confidence bands: *conservative* for $k > 0$.

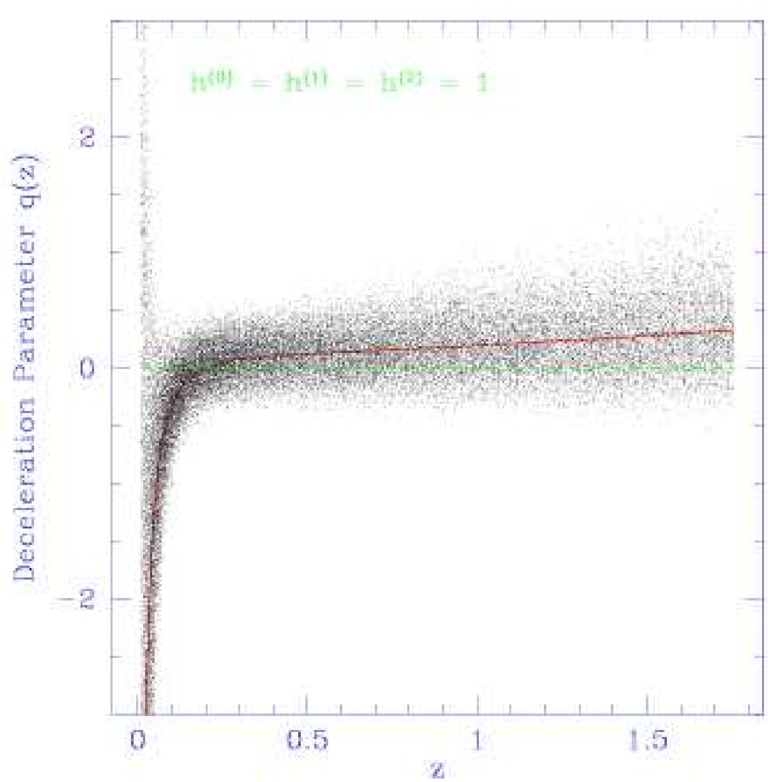
⇒ We estimate 95% confidence bands for $q(z)$ and $w(z)$ by sampling from the bands for $r_l^{(k)}$ (and bounds for $H_0^2\Omega_M$).

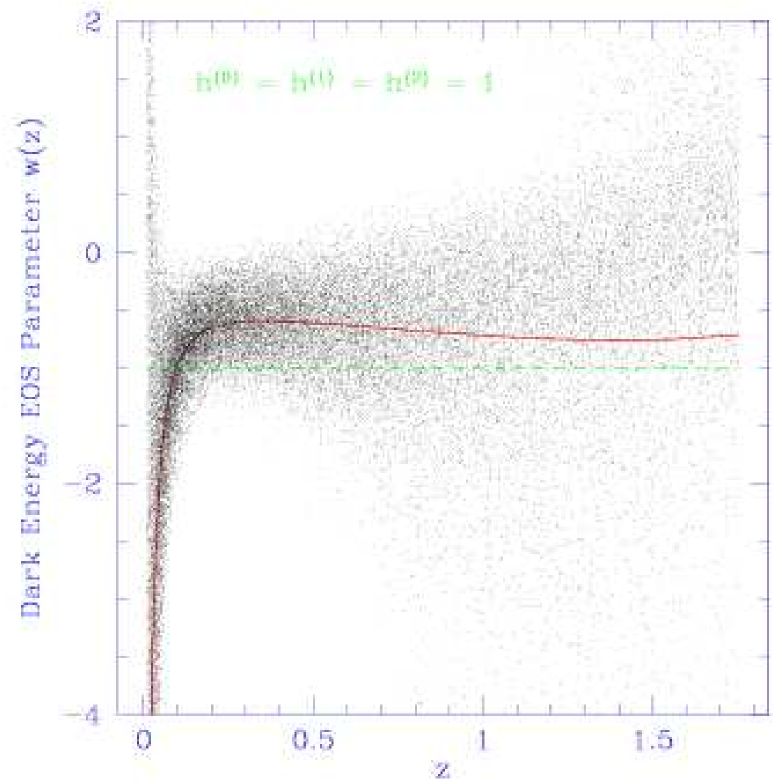






(PRELIMINARY) RESULT: q AND w





CONCLUSIONS

$\Rightarrow$ Local polynomial regression offers a robust mechanism for estimating the function $r_l(z_t)$ and its derivatives. However:

- $\rightarrow$ Bias estimators examined to date are insufficiently accurate to be used to determine optimal bandwidths for function derivatives (given 156 SNe); and
- $\rightarrow$ The Sun & Loader method yields confidence bands that are too small in the high-(h, k, n) limit.

$\Rightarrow$ Preliminary analysis assuming the Sun & Loader method and bandwidths $h^{(k)} = 1$ indicates that at the 95% significance level there are insufficient data:

- $\rightarrow$ to conclude that the universe is accelerating currently; and
- $\rightarrow$ to differentiate between competing theories of dark energy.

$\Rightarrow$ We find these qualitative conclusions to be robust over a large range of $h^{(k)}$ values.

ACKNOWLEDGMENTS

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