Homework 1
Due Thursday September 6
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1. Suppose we toss a fair coin until we get exactly three heads. Describe the sample space \( \Omega \). Let \( X \) denote the number of tosses. Find probability mass function of \( X \).

2. Consider events \( A_1, \ldots, A_n \). Prove that
\[
P \left( \bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} P(A_i).
\]

3. Suppose that \( A \) and \( B \) are independent events. Show that \( A \) and \( B^c \) are independent events.

4. Show that if \( P(A) = 0 \) or \( P(A) = 1 \) then \( A \) is independent of every other event. Show that if \( A \) is independent of itself then \( P(A) \) is either 0 or 1.

5. Let \( X \) have CDF \( F \). Find the CDF of \( Y = \min\{0, X\} \).

6. A random variable \( X \) is stochastically greater than a random variable \( Y \) if \( F_X(t) \leq F_Y(t) \) for all \( t \) and \( F_X(t) < F_Y(t) \) for some \( t \). Prove that, in this case,
\[
P(X > t) \geq P(Y > t) \quad \text{for every } t,
\]
and
\[
P(X > t) > P(Y > t) \quad \text{for some } t.
\]

7. Define
\[
F(x) = \begin{cases} 
0 & x \leq 2 \\
\frac{x-2}{2} & 2 \leq x \leq 4 \\
1 & x > 4.
\end{cases}
\]
Prove that \( F \) is a valid CDF. Find the probability density function.

8. The uniform distribution on \([-3, 3]\) has density:
\[
f_X(x) = \frac{1}{6} \quad \text{for } x \in [-3, 3].
\]
Suppose \( X \) has this density.
(a) Find \( P(X = 1) \).
(b) Find \( P(0.5 \leq X \leq 1.5) \).
(c) Find the cdf of \( Y = X^2 \).

9. Let \((X, Y)\) have a uniform distribution on the unit circle in the plane.
   (a) Show that \( X \) and \( Y \) are not independent.
   (b) Find \( P(X^2 + Y^2 < 1/4) \).

10. Let \( Z_1, \ldots, Z_n \sim N(\mu, \sigma^2) \). Let \( T = \sum_{i=1}^{n} Z_i^2 \). Find \( \mathbb{E}[T] \) and \( \text{Var}(T) \).