Homework 11
36-705
Due: Thursday November 12th by 3pm.

1. Suppose we randomize \( n \) people to treatment or control. The observed data are \((A_1, Y_1), \ldots, (A_n, Y_n)\) where \( A_i \) is binary (\( A_i = 0 \) means control and \( A_i = 1 \) means treatment). We have \( P(A_i = 0) = P(A_i = 1) = 1/2 \). We want to estimate the average treatment effect:

\[
\tau = E[Y(1) - Y(0)]
\]

where \( Y(0) \) and \( Y(1) \) are the potential outcomes (counterfactuals). Let

\[
\hat{\tau}_1 = \frac{1}{n_1} \sum_{i: A_i = 1} Y_i - \frac{1}{n_0} \sum_{i: A_i = 0} Y_i.
\]

Note that \( n_1 \) and \( n_0 \) are random.

(a) Find the limiting distribution of \( \sqrt{n}(\hat{\tau} - \tau) \).

(b) Construct an asymptotic \( 1 - \alpha \) confidence interval for \( \tau \).

2. Suppose we have an observational study with data \((X_1, A_1, Y_1), \ldots, (X_n, A_n, Y_n)\) where \( A_i \) is a binary treatment.

\[ A \perp \perp (Y(1), Y(0)) | X. \]

(a) Suppose that the propensity scores \( \pi(X) = P(A = 1|X) \) is known. Also assume that

\[ 0 < \delta \leq P(A = 1|X) \leq 1 - \delta < 1 \]

for some \( \delta \). A natural estimator then is the Horvitz-Thompson/Inverse-Propensity Score estimator:

\[
\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_i A_i}{\hat{\pi}(X_i)} - \frac{Y_i (1 - A_i)}{1 - \hat{\pi}(X_i)} \right].
\]

Find the limiting distribution of \( \sqrt{n}(\hat{\tau} - \tau) \).

(b) Construct an asymptotic \( 1 - \alpha \) confidence interval for \( \tau \).

(c) Suppose we don’t know \( \pi(x) \) but we have an estimate \( \hat{\pi} \) and that

\[
\sup_x |\hat{\pi}(x) - \pi(x)| \leq \epsilon_n
\]

where \( \epsilon_n \to 0 \). Let

\[
\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_i A_i}{\hat{\pi}(X_i)} - \frac{Y_i (1 - A_i)}{1 - \hat{\pi}(X_i)} \right].
\]

Show that \( \hat{\tau} \xrightarrow{P} \tau \).

3. Suppose we have an observational study with data \((X_1, A_1, Y_1), \ldots, (X_n, A_n, Y_n)\) where \( A_i \in \mathbb{R} \). Assume that

\[ A \perp \perp (Y(a): a \in \mathbb{R}) | X. \]

In this case, we showed that

\[
\psi(a) \equiv E[Y(a)] = \int \mu(x, a)p(x)dx
\]

where \( \mu(x, a) = E[Y|X = x, A = a] \).
(a) Suppose that $\mu(x, a)$ is known. Let

$$ \hat{\psi}(a) = \frac{1}{n} \sum_i \hat{\mu}(X_i, a). $$

Find the limiting distribution of $\sqrt{n}(\hat{\psi}(a) - \psi(a))$.

(b) Suppose that $\mu(x, a)$ is unknown but that $p(a|x)$ is known. Fix $\epsilon > 0$ and define

$$ \hat{\psi}_\epsilon(a) = \frac{1}{2n\epsilon} \sum_i Y_i I(|A_i - a| < \epsilon) \frac{p(A_i|X_i)}{p(A_i|X_i)} $$

where $I(|A_i - a| < \epsilon) = 1$ if $|A_i - a| < \epsilon$ and $I(|A_i - a| < \epsilon) = 0$ otherwise.

Find the mean of $\hat{\psi}_\epsilon(a)$. Let’s denote the mean by $\psi_\epsilon(a)$. Show that $\psi_\epsilon(a) \to \psi(a)$ as $\epsilon \to 0$ (assuming $\mu(x, a)$ is sufficiently smooth).