Homework 4
36-705
Due: Thursday September 24 by 3pm

1. Recall that the Rademacher complexity for a class of functions is

\[ R_n(\mathcal{F}) = \mathbb{E}_{\epsilon,X} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i)\epsilon_i \right|. \]

Let

\[ \mathcal{F} = \left\{ f : f(x) = \langle \beta, x \rangle, \|\beta\|_2 \leq B \right\}. \]

Suppose that each \( X_i \sim N(0, I_d) \) (multivariate Normal). Now show that:

\[ R_n(\mathcal{F}) \leq B \sqrt{\frac{d}{n}}. \]

**Hint:** From Jensen’s inequality: \( \mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]} \).

2. Suppose that we take a collection of sets \( \mathcal{A} \), and a collection of sets \( \mathcal{B} \), and define \( \mathcal{C} \) as:

\[ \mathcal{C} = \{ A \cup B : A \in \mathcal{A}, B \in \mathcal{B} \}. \]

Show that the shattering number:

\[ s(\mathcal{C}, n) \leq s(\mathcal{A}, n) \times s(\mathcal{B}, n). \]

3. Suppose instead of taking the union of individual sets, we simply collected all sets to define:

\[ \mathcal{C} = \{ A : A \in \mathcal{A} \text{ or } A \in \mathcal{B} \}. \]

Show that the shattering number:

\[ s(\mathcal{C}, n) \leq s(\mathcal{A}, n) + s(\mathcal{B}, n). \]

4. Let \( p_\theta \) be the density on \( \mathbb{R}^2 \) that is uniform on a disc of radius \( \theta \). Let \( X_1, \ldots, X_n \sim p_\theta \).

   (a) Write down the likelihood function.
   
   (b) Find a minimal sufficient statistic.
   
   (c) Show that \( X_1 \) is not a sufficient statistic.

5. Define a partition of \( \mathbb{R}^n \) as follows. Two vectors \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \) are in the same element of the partition if and only if \( L(\theta; x_1, \ldots, x_n) \propto L(\theta; y_1, \ldots, y_n) \). Show that this defines a minimal sufficient partition.