1. Let $X$ have mean $\mu$. We say that $X$ is sub-Gaussian if there exists $\sigma^2$ such that
\[
\log \left( \mathbb{E}[e^{t(X-\mu)}] \right) \leq \frac{t^2\sigma^2}{2}
\]
for all $t$.
(i) Show that $X$ is sub-Gaussian if and only if $-X$ is sub-Gaussian.
(ii) Show that if $X$ is sub-Gaussian then
\[
P(X - \mu \geq t) \leq e^{-t^2/(2\sigma^2)}.
\]
(iii) Suppose that $X$ has mean $\mu$ and is sub-Gaussian. Also suppose that $Y$ has mean $\nu$ and is sub-Gaussian. Further, suppose that $X$ and $Y$ are independent. Show that $X + Y$ is sub-Gaussian.

2. Let $X_1, \ldots, X_n$ be iid, with mean $\mu$, $\text{Var}(X_i) = \sigma^2$ and $|X_i| \leq c$. Bernstein’s inequality says that
\[
P(|X_n - \mu| > t) \leq 2 \exp \left\{ -\frac{nt^2}{2\sigma^2 + 2ct/3} \right\}.
\]
Suppose that $X_i$ has a bounded density $p$ supported on $[-1, 1]$. Let $A_n = [-1/n^2, 1/n^2]$. Let $Y_i = I(X_i \in A_n)$. (Here, $I(\cdot)$ is the indicator function.) Use both Hoeffding’s inequality and Bernstein’s inequality to get bounds on
\[
P(\sum_{i=1}^{n} Y_i - \theta_n > t)
\]
where $\theta_n = P(X_i \in A_n)$. Which bound is tighter?

3. Prove or disprove the following:
   (i) If $X_n = O_P(a_n)$ and $Y_n = O(b_n)$ then $X_nY_n = O_P(a_nb_n)$.
   (ii) If $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n + Y_n = O_P(\max\{a_n, b_n\})$.
   (iii) If $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_nY_n = o_P(a_nb_n)$.
   (iv) If $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n + Y_n = O_P(a_n + b_n)$.
   (v) If $X_n = o_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n/Y_n = O_P(a_n/b_n)$.
   (vi) If $X_n = O_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_nY_n = o_P(a_nb_n)$.

4. Let $X$ have continuous cdf $F$. Show that $F(X) \sim \text{Unif}(0, 1)$.

5. Suppose that $X$ and $Y$ are independent. Let $g$ and $h$ be functions. Show that $f(X)$ and $g(Y)$ are independent.
6. **Optional Bonus Question.** For a finite set of points $A \subset \mathbb{R}^2$, the TSP (travelling salesman problem) is to find the shortest possible route that visits each point exactly once and returns to the original points. We denote the length of TSP path by $L(A)$, or formally,

$$L(\{x_1, \ldots, x_n\}) = \min_{\sigma \in S_n} |x_{\sigma(1)} - x_{\sigma(2)}| + |x_{\sigma(2)} - x_{\sigma(3)}| + \cdots + |x_{\sigma(n-1)} + x_{\sigma(n)}| + |x_{\sigma(n)} - x_{\sigma(1)}|,$$

where $S_n$ is the set of bijective functions $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. Let $L(x_1, \ldots, x_n)$ be shorthand for $L(\{x_1, \ldots, x_n\})$. Let $X_1, \ldots, X_n$ be iid from uniform distribution on $[0, 1]^2$.

Our goal is to bound

$$\mathbb{P}(|L(X_1, \ldots, X_n) - \mathbb{E}[L(X_1, \ldots, X_n)]| > \epsilon).$$

In what follows, you may use the following hint: For any $n$ points $x_1, \ldots, x_n$ and $1 \leq i \leq n - 1$,

$$L(\{x_1, \ldots, x_n\}\{x_i\}) \leq L(x_1, \ldots, x_n) \leq L(\{x_1, \ldots, x_n\}\{x_i\}) + 2 \min_{i < j \leq n} \|x_i - x_j\|.$$

(a) For $1 \leq i \leq n$, let $\tilde{X}_i$ be a random variable uniform on $[0, 1]^2$ that is independent of $X_1, \ldots, X_n$, and define

$$Y_i = \mathbb{E}\left[ L(X_1, \ldots, X_i, \tilde{X}_i, \ldots, X_n)|X_1, \ldots, X_i \right].$$

Using the hint, show that for $1 \leq i \leq n - 1$,

$$|Y_i| \leq 2 \max \left\{ g_{n-i}(X_i), \mathbb{E}\left[ g_{n-i}(\tilde{X}_i) \right] \right\},$$

where

$$g_m(x) = \mathbb{E}\left( \min_{1 \leq i \leq m} \|x - X_i\| \right).$$

(b) It is known that $g_m(x) \leq \sqrt{\frac{\pi}{m}}$ and $|Y_n| \leq 2\sqrt{2}$. Use the fact that $L(X_1, \ldots, X_n) - \mathbb{E}[L(X_1, \ldots, X_n)] = \sum_{i=1}^{n} Y_i$, and apply McDiarmid’s inequality to show

$$\mathbb{P}(|L(X_1, \ldots, X_n) - \mathbb{E}[L(X_1, \ldots, X_n)]| > \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2}{4 + 2\pi \log n} \right).$$
