TEST 2
36-707

NAME: Solutions

Question	Score	Out of:
1		25
2		25
3		25
4		25
Total		100

(1) **Let** $(X_1, Y_1), \ldots, (X_n, Y_n)$ **be** n **data points and suppose that**

$$Y_i = r(X_i) + \epsilon_i$$

where $\mathbb{E}(\epsilon_i) = 0$ **and** $\mathrm{Var}(\epsilon_i) = \sigma^2$. **Consider the following estimator:**

$$\widehat{r}(x) = \begin{cases} Y_i & \text{if } x = X_i, \quad i = 1, \ldots, n \\ \overline{Y} & \text{otherwise} \end{cases}$$

where $\overline{Y} = n^{-1} \sum_{i=1}^n Y_i$.

(a) **Find the weights** $\ell(x) = (\ell_1(x), \ldots, \ell_n(x))$ **so that** $\widehat{r}(x) = \sum_{i=1}^n \ell_i(x)$.

If $x = x_j$ for $j = 1, \ldots, n$

$$\ell_i(x) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$x \neq x_j$ for any $j = 1, \ldots, n$, $\ell_i(x) = 1/n$

(b) **Find the smoothing matrix** L.

$L_{ij} = \ell_j(x_i) \Rightarrow L = I_n$

(c) **Find the effective degrees of freedom.**

$$\nu = tr(L) = tr(I_n) = n$$

(d) **Find the bias and variance. (There are two cases. Case 1:** $x \in \{X_1, \ldots, X_n\}$. **Case 2:** $x \notin \{X_1, \ldots, X_n\}$.)

<u>Case 1</u> Then $\hat{r}(x) = \hat{r}(X_i)$ for some i

$$\text{bias} = E[\hat{r}(x)] - r(x) = E[Y_i] - r(x) = r(x_i) - r(x) = \boxed{0}$$

$$\text{var} = \cancel{V}(\hat{r}(x)) = V(Y_i) = V(\varepsilon_i) = \boxed{\sigma^2}$$

<u>Case 2</u> $\hat{r}(x) \neq \hat{r}(X_i)$ for some i.

$$\text{bias} = E[\hat{r}(x)] - r(x) = E[\bar{Y}] - r(x)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[Y_i] - r(x)$$

$$= \boxed{\frac{1}{n}\sum_{i=1}^{n} r(X_i) - r(x)}$$

$$\text{var}(\hat{r}(x)) = \text{Var}(\bar{Y}) = \boxed{\sigma^2/n}$$

(2) Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be n data points and let

$$\widehat{r}(x) = \frac{\sum_{i=1}^n Y_i K\left(\frac{x-X_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-X_i}{h}\right)}$$

be the kernel regression estimator where $K(z) = e^{-z^2/2}$ is the Gaussian kernel.

(a) Show that $\widehat{r}(X_i) \to Y_i$ as $h \to 0$.

$$\hat{r}(x_j) = \frac{\sum_{i=1}^n y_i K\left(\frac{x_j - x_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x_j - x_i}{h}\right)}$$

$$= \frac{\sum_{i=1}^n y_i e^{-\left(\frac{x_j - x_i}{h}\right)^2/2}}{\sum_{i=1}^n e^{-\left(\frac{x_j - x_i}{h}\right)^2/2}}$$

but $x_j - x_i = 0$ when $j = i \Rightarrow K(0) = 1$

$$= \frac{y_j + \sum_{i=1, i \neq j}^n y_i e^{-\left(\frac{x_j - x_i}{h}\right)^2/2}}{1 + \sum_{i=1, i \neq j}^n e^{-\left(\frac{x_j - x_i}{h}\right)^2/2}}$$

As $h \to 0$, $\frac{x_j - x_i}{h} \longrightarrow \infty \Rightarrow K(\infty) = e^{-\infty} = 0$

$$\therefore = \frac{y_j + \sum_{i=1, i \neq j}^n y_i \cdot 0}{1 + 0} = \frac{y_j}{1} = y_j$$

$\therefore \hat{r}(x_j) \longrightarrow y_j$ as $h \to 0$

(b) Show that $\widehat{r}(X_i) \to \overline{Y}$ as $h \to \infty$.

$$\hat{r}(x_i) = \frac{\sum_{j=1}^{n} Y_j e^{-\left(\frac{X_i - X_j}{h}\right)^2/2}}{\sum_{j=1}^{n} e^{-\left(\frac{X_i - X_j}{h}\right)^2/2}}$$

As $h \to \infty$, $\frac{x_i - x_j}{h} \to 0 \Rightarrow K(0) = 1$

$$\therefore \quad = \frac{\sum_{j=1}^{n} Y_j \cdot 1}{\sum_{j=1}^{n} 1} = \frac{\sum_{j=1}^{n} Y_j}{n} = \overline{Y}$$

(3) Suppose that $Y \in \{0, 1\}$ and $\mathbb{P}(Y = 1) = 1/2$. The distribution of $X|Y = 0$ is discrete and is given by

$$\mathbb{P}(X = 1|Y = 0) = \mathbb{P}(X = 2|Y = 0) = 1/2.$$

The distribution of $X|Y = 1$ is discrete and is given by

$$\mathbb{P}(X = 2|Y = 1) = \mathbb{P}(X = 3|Y = 1) = 1/2.$$

Find the Bayes rule and the Bayes risk.

Bayes Rule: $h^*(x) = \begin{cases} 1 & r(x) > 1/2 \\ 0 & \text{ow} \end{cases}$

$= \begin{cases} 1 & P(Y=1 \mid X=x) > 1/2 \\ 0 & \text{ow} \end{cases}$

$= \begin{cases} 1 & X=3 \\ 0 & \text{ow} \end{cases}$

Risk:

$R^*(x) = P(h^*(x) \neq Y)$

$= P(h^*(x) \neq Y \mid X=1)\, P(X=1) + P(h^*(x) \neq Y \mid X=2) \cdot P(X=2)$

$+ P(h^*(x) \neq Y \mid X=3)\, P(X=3)$

$= 0 + \frac{1}{2} P(X=2) + 0$

$= \frac{1}{2} \left(P(X=2 \mid Y=0) P(Y=0) + P(X=2 \mid Y=1) P(Y=1) \right)$

$= \frac{1}{2} \left(\frac{1}{2} \right) = \boxed{\frac{1}{4}}$

Assume $P(Y=1) = \pi$.

(4) Let $Y \in \{0,1\}$, $f_0(x) = f(x|Y=0)$ and $f_1(x) = f(x|Y=1)$. Suppose that $f_0 = f_1$. Find the Bayes rule and the Bayes risk.

$$h^*(x) = \begin{cases} 1 & \dfrac{f_1(x)}{f_0(x)} > \dfrac{1-\pi}{\pi} \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1 & 1 > \frac{1-\pi}{\pi} \\ 0 & \text{ow} \end{cases}$$

$$= \begin{cases} 1 & \pi > 1/2 \\ 0 & \text{ow} \end{cases}$$

Risk:

$R^* = P(h^* \neq Y) = P(h^*=0, Y=1) + P(h^*=1, Y=0)$

Case 1: $\pi > 1/2$

$$R^* = 0 + (1-\pi)$$

Case 2: $\pi \leq 1/2$

$$R^* = \pi + 0$$

$$R^* = \min(1-\pi, \pi)$$