Last time: duality in linear programs

Given \( c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, G \in \mathbb{R}^{r \times n}, h \in \mathbb{R}^r: \)

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad Gx \leq h
\end{align*}
\]

Primal LP

\[
\begin{align*}
\max_{u \in \mathbb{R}^m, v \in \mathbb{R}^r} & \quad -b^T u - h^T v \\
\text{subject to} & \quad -A^T u - G^T v = c \\
& \quad v \geq 0
\end{align*}
\]

Dual LP

Explanation: for any \( u \) and \( v \geq 0 \), and \( x \) primal feasible,

\[
u^T (Ax - b) + v^T (Gx - h) \leq 0, \quad \text{i.e.,} \\
(-A^T u - G^T v)^T x \geq -b^T u - h^T v
\]

So if \( c = -A^T u - G^T v \), we get a bound on primal optimal value
Explanation # 2: for any $u$ and $v \geq 0$, and $x$ primal feasible

$$c^T x \geq c^T x + u^T (Ax - b) + v^T (Gx - h) := L(x, u, v)$$

So if $C$ denotes primal feasible set, $f^*$ primal optimal value, then for any $u$ and $v \geq 0$,

$$f^* \geq \min_{x \in C} L(x, u, v) \geq \min_x L(x, u, v) := g(u, v)$$

In other words, $g(u, v)$ is a lower bound on $f^*$ for any $u$ and $v \geq 0$. Note that

$$g(u, v) = \begin{cases} -b^T u - h^T v & \text{if } c = -A^T u - G^T v \\ -\infty & \text{otherwise} \end{cases}$$

This second explanation reproduces the same dual, but is actually completely general and applies to arbitrary optimization problems (even nonconvex ones)
Today:

- Lagrange dual function
- Langrange dual problem
- Weak and strong duality
- Examples
- Preview of duality uses
Consider general minimization problem

\[
\begin{align*}
\min & \quad f(x) \\
\text{subject to} & \quad h_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad \ell_j(x) = 0, \quad j = 1, \ldots, r
\end{align*}
\]

Need not be convex, but of course we will pay special attention to convex case

We define the \textbf{Lagrangian} as

\[
L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x)
\]

New variables \( u \in \mathbb{R}^m, v \in \mathbb{R}^r \), with \( u \geq 0 \) (implicitly, we define \( L(x, u, v) = -\infty \) for \( u < 0 \))
Important property: for any $u \geq 0$ and $v$,

$$f(x) \geq L(x, u, v)$$

at each feasible $x$

Why? For feasible $x$,

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x) \leq f(x)$$

- Solid line is $f$
- Dashed line is $h$, hence feasible set $\approx [-0.46, 0.46]$
- Each dotted line shows $L(x, u, v)$ for different choices of $u \geq 0$ and $v$

(From B & V page 217)
Lagrange dual function

Let $C$ denote primal feasible set, $f^*$ denote primal optimal value. Minimizing $L(x, u, v)$ over all $x$ gives a lower bound:

$$f^* \geq \min_{x \in C} L(x, u, v) \geq \min_x L(x, u, v) := g(u, v)$$

We call $g(u, v)$ the Lagrange dual function, and it gives a lower bound on $f^*$ for any $u \geq 0$ and $v$, called dual feasible $u, v$

- Dashed horizontal line is $f^*$
- Dual variable $\lambda$ is (our $u$)
- Solid line shows $g(\lambda)$

(From B & V page 217)
Example: quadratic program

Consider quadratic program:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x$$

subject to \( Ax = b, \ x \geq 0 \)

where \( Q \succ 0 \). Lagrangian:

$$L(x, u, v) = \frac{1}{2} x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

Lagrange dual function:

$$g(u, v) = \min_{x \in \mathbb{R}^n} L(x, u, v) = -\frac{1}{2} (c-u+ATv)^T Q^{-1} (c-u+ATv) - b^T v$$

For any \( u \geq 0 \) and any \( v \), this is lower a bound on primal optimal value \( f^* \)
Same problem

\[
\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T Q x + c^T x
\]
subject to \( Ax = b, \ x \geq 0 \)

but now \( Q \succeq 0 \). Lagrangian:

\[
L(x, u, v) = \frac{1}{2} x^T Q x + c^T x - u^T x + v^T (Ax - b)
\]

Lagrange dual function:

\[
g(u, v) = \begin{cases} 
-\frac{1}{2} (c - u + A^T v)^T Q^+ (c - u + A^T v) - b^T v \\
-\infty
\end{cases}
\]

if \( c - u + A^T v \perp \text{null}(Q) \)

otherwise

where \( Q^+ \) denotes generalized inverse of \( Q \). For any \( u \geq 0, v, \) and \( c - u + A^T v \perp \text{null}(Q) \), \( g(u, v) \) is a nontrivial lower bound on \( f^* \)
Example: quadratic program in 2D

We choose $f(x)$ to be quadratic in 2 variables, subject to $x \geq 0$. Dual function $g(u)$ is also quadratic in 2 variables, also subject to $u \geq 0$

Dual function $g(u)$ provides a bound on $f^*$ for every $u \geq 0$

Largest bound this gives us: turns out to be exactly $f^*$ ... coincidence?

More on this later, via KKT conditions
Lagrange dual problem

Given primal problem

\[
\begin{align*}
\min_x & \quad f(x) \\
\text{subject to} & \quad h_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad \ell_j(x) = 0, \quad j = 1, \ldots, r
\end{align*}
\]

Our constructed dual function \( g(u, v) \) satisfies \( f^* \geq g(u, v) \) for all \( u \geq 0 \) and \( v \). Hence best lower bound is given by maximizing \( g(u, v) \) over all dual feasible \( u, v \), yielding Lagrange dual problem:

\[
\begin{align*}
\max_{u, v} & \quad g(u, v) \\
\text{subject to} & \quad u \geq 0
\end{align*}
\]

Key property, called weak duality: if dual optimal value \( g^* \), then

\[
f^* \geq g^*
\]

Note that this always holds (even if primal problem is nonconvex)
Another key property: the dual problem is a **convex optimization** problem (as written, it is a concave maximization problem)

Again, this is always true (even when primal problem is not convex)

By definition:

\[
g(u, v) = \min_x \left\{ f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x) \right\}
\]

\[
= -\max_x \left\{ -f(x) - \sum_{i=1}^{m} u_i h_i(x) - \sum_{j=1}^{r} v_j \ell_j(x) \right\}
\]

pointwise maximum of convex functions in \((u, v)\)

I.e., \(g\) is concave in \((u, v)\), and \(u \geq 0\) is a convex constraint, hence dual problem is a concave maximization problem
Example: nonconvex quartic minimization

Define $f(x) = x^4 - 50x^2 + 100x$ (nonconvex), minimize subject to constraint $x \geq -4.5$

Dual function $g$ can be derived explicitly (via closed-form equation for roots of a cubic equation). Form of $g$ is quite complicated, and would be hard to check directly if $g$ is concave ... but it must be!
Strong duality

Recall that we always have $f^* \geq g^*$ (weak duality). On the other hand, in some problems we have observed that actually

$$f^* = g^*$$

which is called strong duality

Slater’s condition: if the primal is a convex problem (i.e., $f$ and $h_1, \ldots, h_m$ are convex, $\ell_1, \ldots, \ell_r$ are affine), and there exists at least one strictly feasible $x \in \mathbb{R}^n$, meaning

$$h_1(x) < 0, \ldots, h_m(x) < 0 \quad \text{and} \quad \ell_1(x) = 0, \ldots, \ell_r(x) = 0$$

then strong duality holds

This is a pretty weak condition. (Further refinement: only require strict inequalities over functions $h_i$ that are not affine)
LPs: back to where we started

For linear programs:

- Easy to check that the dual of the dual LP is the primal LP
- Refined version of Slater’s condition: strong duality holds for an LP if it is feasible
- Apply same logic to its dual LP: strong duality holds if it is feasible
- Hence strong duality holds for LPs, except when both primal and dual are infeasible

In other words, we nearly always have strong duality for LPs
Example: support vector machine dual

Given \( y \in \{-1, 1\}^n, X \in \mathbb{R}^{n \times p} \), rows \( x_1, \ldots, x_n \), recall the support vector machine problem:

\[
\min_{\beta, \beta_0, \xi} \quad \frac{1}{2} \|\beta\|_2^2 + C \sum_{i=1}^{n} \xi_i \\
\text{subject to} \quad \xi_i \geq 0, \ i = 1, \ldots, n \\
y_i(x_i^T \beta + \beta_0) \geq 1 - \xi_i, \ i = 1, \ldots, n
\]

Introducing dual variables \( v, w \geq 0 \), we form the Lagrangian:

\[
L(\beta, \beta_0, \xi, v, w) = \frac{1}{2} \|\beta\|_2^2 + C \sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} v_i \xi_i + \\
\sum_{i=1}^{n} w_i (1 - \xi_i - y_i(x_i^T \beta + \beta_0))
\]
Minimizing over $\beta, \beta_0, \xi$ gives Lagrange dual function:

$$g(v, w) = \begin{cases} -\frac{1}{2} w^T \tilde{X} \tilde{X}^T w + 1^T w & \text{if } w = C1 - v, \ w^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

where $\tilde{X} = \text{diag}(y)X$. Thus SVM dual problem, eliminating slack variable $v$, becomes

$$\max_w \ -\frac{1}{2} w^T \tilde{X} \tilde{X}^T w + 1^T w$$

subject to $0 \leq w \leq C1, \ w^T y = 0$

Check: Slater’s condition is satisfied, and we have strong duality. Further, from study of SVMs, might recall that at optimality

$$\beta = \tilde{X}^T w$$

This is not a coincidence, as we’ll later via the KKT conditions
Duality gap

Given primal feasible $x$ and dual feasible $u, v$, the quantity

$$f(x) - g(u, v)$$

is called the **duality gap** between $x$ and $u, v$. Note that

$$f(x) - f^* \leq f(x) - g(u, v)$$

so if the duality gap is zero, then $x$ is primal optimal (and similarly, $u, v$ are dual optimal)

From an algorithmic viewpoint, provides a stopping criterion: if $f(x) - g(u, v) \leq \epsilon$, then we are guaranteed that $f(x) - f^* \leq \epsilon$

Very useful, especially in conjunction with iterative methods ... more dual uses in coming lectures
Dual norms

Let \( \| x \| \) be a norm, e.g.,

- \( \ell_p \) norm: \( \| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \), for \( p \geq 1 \)
- Trace norm: \( \| X \|_{tr} = \sum_{i=1}^{r} \sigma_i(X) \)

We define its dual norm \( \| x \|_* \) as

\[
\| x \|_* = \max_{\| z \| \leq 1} z^T x
\]

Gives us the inequality \( |z^T x| \leq \| z \| \| x \|_* \), like Cauchy-Schwartz.

Back to our examples,

- \( \ell_p \) norm dual: \( (\| x \|_p)_* = \| x \|_q \), where \( 1/p + 1/q = 1 \)
- Trace norm dual: \( (\| X \|_{tr})_* = \| X \|_{op} = \sigma_{\max}(X) \)

Dual norm of dual norm: it turns out that \( \| x \|_{**} = \| x \| \) ... we’ll see connections to duality (including this one) in coming lectures
References