Convexity I: Sets and Functions

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Convex Optimization 10-725

See supplements for reviews of

- *basic real analysis*
- *basic multivariate calculus*
- *basic linear algebra*

Last time: why convexity?

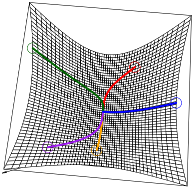
Why convexity? Simply put: because we can broadly understand and solve convex optimization problems

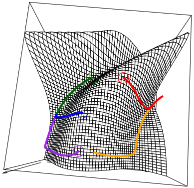
Nonconvex problems are mostly treated on a case by case basis

Reminder: a convex optimization problem is of the form

$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \; i = 1, \ldots, m \\ & h_j(x) = 0, \; j = 1, \ldots, r \end{aligned}$$

where f and g_i, $i = 1, \ldots, m$ are all convex, and h_j, $j = 1, \ldots, r$ are affine. Special property: any local minimizer is a global minimizer





Outline

Today:

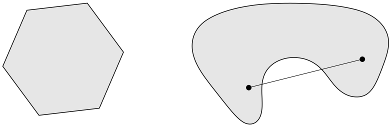
- Convex sets
- Examples
- Key properties
- Operations preserving convexity
- Same, for convex functions

Convex sets

Convex set: $C \subseteq \mathbb{R}^n$ such that

$$x, y \in C \implies tx + (1-t)y \in C \text{ for all } 0 \leq t \leq 1$$

In words, line segment joining any two elements lies entirely in set



Convex combination of $x_1, \ldots, x_k \in \mathbb{R}^n$: any linear combination

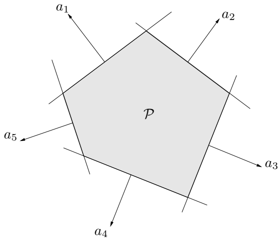
$$\theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_i \geq 0$, $i = 1, \ldots, k$, and $\sum_{i=1}^k \theta_i = 1$. Convex hull of a set C, $\mathrm{conv}(C)$, is all convex combinations of elements. Always convex

Examples of convex sets

- Trivial ones: empty set, point, line
- Norm ball: $\{x : \|x\| \leq r\}$, for given norm $\|\cdot\|$, radius r
- Hyperplane: $\{x : a^T x = b\}$, for given a, b
- Halfspace: $\{x : a^T x \leq b\}$
- Affine space: $\{x : Ax = b\}$, for given A, b

- Polyhedron: $\{x : Ax \leq b\}$, where inequality $\leq$ is interpreted componentwise. Note: the set $\{x : Ax \leq b, Cx = d\}$ is also a polyhedron (why?)



- Simplex: special case of polyhedra, given by $\mathrm{conv}\{x_0, \ldots, x_k\}$, where these points are affinely independent. The canonical example is the probability simplex,

$$\mathrm{conv}\{e_1, \ldots, e_n\} = \{w : w \geq 0,\ 1^T w = 1\}$$

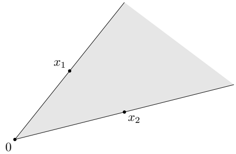
Cones

Cone: $C \subseteq \mathbb{R}^n$ such that

$$x \in C \implies tx \in C \text{ for all } t \geq 0$$

Convex cone: cone that is also convex, i.e.,

$$x_1, x_2 \in C \implies t_1 x_1 + t_2 x_2 \in C \text{ for all } t_1, t_2 \geq 0$$



Conic combination of $x_1, \ldots, x_k \in \mathbb{R}^n$: any linear combination

$$\theta_1 x_1 + \cdots + \theta_k x_k$$

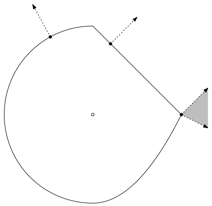
with $\theta_i \geq 0$, $i = 1, \ldots, k$. Conic hull collects all conic combinations

Examples of convex cones

- Norm cone: $\{(x,t) : \|x\| \le t\}$, for a norm $\|\cdot\|$. Under the ℓ_2 norm $\|\cdot\|_2$, called second-order cone

- Normal cone: given any set C and point $x \in C$, we can define

$$\mathcal{N}_C(x) = \{g : g^T x \ge g^T y, \text{ for all } y \in C\}$$

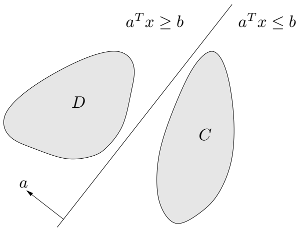


This is always a convex cone, regardless of C

- Positive semidefinite cone: $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : X \succeq 0\}$, where $X \succeq 0$ means that X is positive semidefinite (and $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices)

Key properties of convex sets

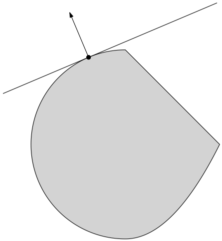
- Separating hyperplane theorem: two disjoint convex sets have a separating between hyperplane them



Formally: if C, D are nonempty convex sets with $C \cap D = \emptyset$, then there exists a, b such that

$$C \subseteq \{x : a^T x \leq b\}$$
$$D \subseteq \{x : a^T x \geq b\}$$

- Supporting hyperplane theorem: a boundary point of a convex set has a supporting hyperplane passing through it



Formally: if C is a nonempty convex set, and $x_0 \in \mathrm{bd}(C)$, then there exists a such that

$$C \subseteq \{x : a^T x \leq a^T x_0\}$$

Both of the above theorems (separating and supporting hyperplane theorems) have partial converses; see Section 2.5 of BV

Operations preserving convexity

- Intersection: the intersection of convex sets is convex
- Scaling and translation: if C is convex, then

  $$aC + b = \{ax + b : x \in C\}$$

  is convex for any a, b
- Affine images and preimages: if $f(x) = Ax + b$ and C is convex then

  $$f(C) = \{f(x) : x \in C\}$$

  is convex, and if D is convex then

  $$f^{-1}(D) = \{x : f(x) \in D\}$$

  is convex

Example: linear matrix inequality solution set

Given $A_1, \ldots, A_k, B \in \mathbb{S}^n$, a linear matrix inequality is of the form

$$x_1 A_1 + x_2 A_2 + \cdots + x_k A_k \preceq B$$

for a variable $x \in \mathbb{R}^k$. Let's prove the set C of points x that satisfy the above inequality is convex

Approach 1: directly verify that $x, y \in C \Rightarrow tx + (1-t)y \in C$. This follows by checking that, for any v,

$$v^T \Big(B - \sum_{i=1}^{k} (tx_i + (1-t)y_i) A_i \Big) v \geq 0$$

Approach 2: let $f : \mathbb{R}^k \to \mathbb{S}^n$, $f(x) = B - \sum_{i=1}^k x_i A_i$. Note that $C = f^{-1}(\mathbb{S}^n_+)$, affine preimage of convex set

More operations preserving convexity

- Perspective images and preimages: the perspective function is $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ (where $\mathbb{R}_{++}$ denotes positive reals),

  $$P(x, z) = x/z$$

  for $z > 0$. If $C \subseteq \mathrm{dom}(P)$ is convex then so is $P(C)$, and if D is convex then so is $P^{-1}(D)$

- Linear-fractional images and preimages: the perspective map composed with an affine function,

  $$f(x) = \frac{Ax + b}{c^T x + d}$$

  is called a linear-fractional function, defined on $c^T x + d > 0$. If $C \subseteq \mathrm{dom}(f)$ is convex then so if $f(C)$, and if D is convex then so is $f^{-1}(D)$

Example: conditional probability set

Let U, V be random variables over $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$. Let $C \subseteq \mathbb{R}^{nm}$ be a set of joint distributions for U, V, i.e., each $p \in C$ defines joint probabilities

$$p_{ij} = \mathbb{P}(U = i, V = j)$$

Let $D \subseteq \mathbb{R}^{nm}$ contain corresponding conditional distributions, i.e., each $q \in D$ defines

$$q_{ij} = \mathbb{P}(U = i | V = j)$$

Assume C is convex. Let's prove that D is convex. Write

$$D = \left\{ q \in \mathbb{R}^{nm} : q_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}, \text{ for some } p \in C \right\} = f(C)$$

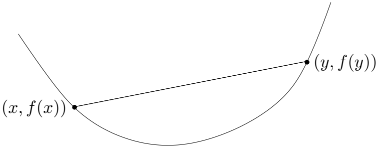
where f is a linear-fractional function, hence D is convex

Convex functions

Convex function: $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ convex, and

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for } 0 \le t \le 1$$

and all $x, y \in \mathrm{dom}(f)$



In words, function lies below the line segment joining $f(x), f(y)$

Concave function: opposite inequality above, so that

$$f \text{ concave} \iff -f \text{ convex}$$

Important modifiers:

- Strictly convex: $f\big(tx + (1-t)y\big) < tf(x) + (1-t)f(y)$ for $x \neq y$ and $0 < t < 1$. In words, f is convex and has greater curvature than a linear function

- Strongly convex with parameter $m > 0$: $f - \frac{m}{2}\|x\|_2^2$ is convex. In words, f is at least as convex as a quadratic function

Note: strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex

(Analogously for concave functions)

Examples of convex functions

- Univariate functions:
  - Exponential function: e^{ax} is convex for any a over $\mathbb{R}$
  - Power function: x^a is convex for $a \geq 1$ or $a \leq 0$ over $\mathbb{R}_+$ (nonnegative reals)
  - Power function: x^a is concave for $0 \leq a \leq 1$ over $\mathbb{R}_+$
  - Logarithmic function: $\log x$ is concave over $\mathbb{R}_{++}$
- Affine function: $a^T x + b$ is both convex and concave
- Quadratic function: $\frac{1}{2} x^T Q x + b^T x + c$ is convex provided that $Q \succeq 0$ (positive semidefinite)
- Least squares loss: $\|y - Ax\|_2^2$ is always convex (since $A^T A$ is always positive semidefinite)

- Norm: $\|x\|$ is convex for any norm; e.g., ℓ_p norms,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{for } p \geq 1, \quad \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

and also operator (spectral) and trace (nuclear) norms,

$$\|X\|_{\mathrm{op}} = \sigma_1(X), \quad \|X\|_{\mathrm{tr}} = \sum_{i=1}^{r} \sigma_r(X)$$

where $\sigma_1(X) \geq \ldots \geq \sigma_r(X) \geq 0$ are the singular values of the matrix X

- Indicator function: if C is convex, then its indicator function

$$I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

  is convex

- Support function: for any set C (convex or not), its support function

$$I_C^*(x) = \max_{y \in C} \; x^T y$$

  is convex

- Max function: $f(x) = \max\{x_1, \ldots, x_n\}$ is convex

Key properties of convex functions

- A function is convex if and only if its restriction to any line is convex
- Epigraph characterization: a function f is convex if and only if its epigraph

$$\mathrm{epi}(f) = \{(x, t) \in \mathrm{dom}(f) \times \mathbb{R} : f(x) \leq t\}$$

  is a convex set
- Convex sublevel sets: if f is convex, then its sublevel sets

$$\{x \in \mathrm{dom}(f) : f(x) \leq t\}$$

  are convex, for all $t \in \mathbb{R}$. The converse is not true

- First-order characterization: if f is differentiable, then f is convex if and only if $\mathrm{dom}(f)$ is convex, and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

  for all $x, y \in \mathrm{dom}(f)$. Therefore for a differentiable convex function $\nabla f(x) = 0 \iff x$ minimizes f

- Second-order characterization: if f is twice differentiable, then f is convex if and only if $\mathrm{dom}(f)$ is convex, and $\nabla^2 f(x) \succeq 0$ for all $x \in \mathrm{dom}(f)$

- Jensen's inequality: if f is convex, and X is a random variable supported on $\mathrm{dom}(f)$, then $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$

Operations preserving convexity

- Nonnegative linear combination: $f_1, \ldots, f_m$ convex implies $a_1 f_1 + \cdots + a_m f_m$ convex for any $a_1, \ldots, a_m \geq 0$
- Pointwise maximization: if f_s is convex for any $s \in S$, then $f(x) = \max_{s \in S}\ f_s(x)$ is convex. Note that the set S here (number of functions f_s) can be infinite
- Partial minimization: if $g(x, y)$ is convex in x, y, and C is convex, then $f(x) = \min_{y \in C}\ g(x, y)$ is convex

Example: distances to a set

Let C be an arbitrary set, and consider the maximum distance to C under an arbitrary norm $\|\cdot\|$:

$$f(x) = \max_{y \in C} \|x - y\|$$

Let's check convexity: $f_y(x) = \|x - y\|$ is convex in x for any fixed y, so by pointwise maximization rule, f is convex

Now let C be convex, and consider the minimum distance to C:

$$f(x) = \min_{y \in C} \|x - y\|$$

Let's check convexity: $g(x, y) = \|x - y\|$ is convex in x, y jointly, and C is assumed convex, so apply partial minimization rule

More operations preserving convexity

- Affine composition: if f is convex, then $g(x) = f(Ax + b)$ is convex

- General composition: suppose $f = h \circ g$, where $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. Then:
  - f is convex if h is convex and nondecreasing, g is convex
  - f is convex if h is convex and nonincreasing, g is concave
  - f is concave if h is concave and nondecreasing, g concave
  - f is concave if h is concave and nonincreasing, g convex

  How to remember these? Think of the chain rule when $n = 1$:

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- Vector composition: suppose that

$$f(x) = h\big(g(x)\big) = h\big(g_1(x), \ldots, g_k(x)\big)$$

where $g : \mathbb{R}^n \to \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. Then:

  - f is convex if h is convex and nondecreasing in each argument, g is convex
  - f is convex if h is convex and nonincreasing in each argument, g is concave
  - f is concave if h is concave and nondecreasing in each argument, g is concave
  - f is concave if h is concave and nonincreasing in each argument, g is convex

Example: log-sum-exp function

Log-sum-exp function: $g(x) = \log(\sum_{i=1}^{k} e^{a_i^T x + b_i})$, for fixed a_i, b_i, $i = 1, \ldots, k$. Often called "soft max", as it smoothly approximates $\max_{i=1,\ldots k}\,(a_i^T x + b_i)$

How to show convexity? First, note it suffices to prove convexity of $f(x) = \log(\sum_{i=1}^{n} e^{x_i})$ (affine composition rule)

Now use second-order characterization. Calculate

$$\nabla_i f(x) = \frac{e^{x_i}}{\sum_{\ell=1}^{n} e^{x_\ell}}$$

$$\nabla_{ij}^2 f(x) = \frac{e^{x_i}}{\sum_{\ell=1}^{n} e^{x_\ell}} 1\{i = j\} - \frac{e^{x_i} e^{x_j}}{(\sum_{\ell=1}^{n} e^{x_\ell})^2}$$

Write $\nabla^2 f(x) = \mathrm{diag}(z) - zz^T$, where $z_i = e^{x_i} / (\sum_{\ell=1}^{n} e^{x_\ell})$. This matrix is diagonally dominant, hence positive semidefinite

References and further reading

- S. Boyd and L. Vandenberghe (2004), "Convex optimization", Chapters 2 and 3
- J.P. Hiriart-Urruty and C. Lemarechal (1993), "Fundamentals of convex analysis", Chapters A and B
- R. T. Rockafellar (1970), "Convex analysis", Chapters 1–10,