Frank-Wolfe Method

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Last time: ADMM

For the problem

$$\min_{x,z} f(x) + g(z) \text{ subject to } Ax + Bz = c$$

we form augmented Lagrangian (scaled form):

$$L_\rho(x, z, w) = f(x) + g(z) + \frac{\rho}{2} \|Ax - Bx + c + w\|_2^2 - \frac{\rho}{2} \|w\|_2^2$$

Alternating direction method of multipliers or ADMM:

$$x^{(k)} = \arg\min_x L_\rho(x, z^{(k-1)}, w^{(k-1)})$$
$$z^{(k)} = \arg\min_z L_\rho(x^{(k)}, z, w^{(k-1)})$$
$$w^{(k)} = w^{(k-1)} + Ax^{(k)} + Bz^{(k)} - c$$

Converges like a first-order method. Very flexible framework
Projected gradient descent

Consider constrained problem

$$\min_x f(x) \text{ subject to } x \in C$$

where $f$ is convex and smooth, and $C$ is convex. Recall **projected gradient descent** chooses an initial $x^{(0)}$, repeats for $k = 1, 2, 3, \ldots$

$$x^{(k)} = P_C(x^{(k-1)} - t_k \nabla f(x^{(k-1)})$$

where $P_C$ is the projection operator onto the set $C$. Special case of proximal gradient, motivated by local quadratic expansion of $f$:

$$x^{(k)} = P_C \left( \arg\min_y \nabla f(x^{(k-1)})^T (y - x^{(k-1)}) + \frac{1}{2t} \| y - x^{(k-1)} \|_2^2 \right)$$

Motivation for today: **projections are not always easy!**
Frank-Wolfe method

The Frank-Wolfe method, also called conditional gradient method, uses a local linear expansion of $f$:

\[
\begin{align*}
    s^{(k-1)} & \in \arg\min_{s \in C} \nabla f(x^{(k-1)})^T s \\
    x^{(k)} & = (1 - \gamma_k)x^{(k-1)} + \gamma_k s^{(k-1)}
\end{align*}
\]

Note that there is no projection; update is solved directly over $C$

Default step sizes: $\gamma_k = 2/(k + 1)$, $k = 1, 2, 3, \ldots$. Note for any $0 \leq \gamma_k \leq 1$, we have $x^{(k)} \in C$ by convexity. Can rewrite update as

\[
    x^{(k)} = x^{(k-1)} + \gamma_k (s^{(k-1)} - x^{(k-1)})
\]

i.e., we are moving less and less in the direction of the linearization minimizer as the algorithm proceeds
A Banach space equipped with an inner product is a compact and convex subset of a Hilbert space. The method (known as iterative optimizers) is given by the Frank-Wolfe optimization problems, one of the simplest and earliest. We assume that the objective function is a compact convex subset of any vector space, continuously differentiable, and that the domain of the objective function is a convex hull of an atomic set, even if those optimal points have inexact solutions. We assume that the objective function is convex and continuously differentiable, and that the domain is a compact convex subset of a Hilbert space. Inexact solutions are obtained by extending the duality concept as well as the primal-dual convergence results for Frank-Wolfe algorithms. We provide stronger and more general convergence analysis for the presented analysis unifies several existing convergence results for different sparse greedy algorithm variants into one simplified proof. In contrast to existing analysis of the Frank-Wolfe algorithm, our convergence analysis for the general Frank-Wolfe algorithm takes into account approximate linear subproblems. Furthermore, our analysis is invariant under any affine transformation/pre-conditioning: At a current position, a step of this algorithm is illustrated in the inset figure: At a current position, a step of this algorithm is illustrated in the inset figure: At a current position, a step of this algorithm is illustrated in the inset figure:

Algorithm 1

\begin{algorithm}
\caption{Frank-Wolfe Algorithm}
\begin{algorithmic}
\FOR {$k = 0, 1, \ldots$}
\STATE Compute $s^k := \text{arg min}_{s \in \mathcal{D}} \langle g(x^k), s \rangle$
\STATE $x^{k+1} := x^k + s^k$
\ENDFOR
\end{algorithmic}
\end{algorithm}

\begin{itemize}
\item $s^k := \text{arg min}_{s \in \mathcal{D}} \langle g(x^k), s \rangle$
\item $x^{k+1} := x^k + s^k$
\end{itemize}

The contributions of this paper are towards a minimizer of the linear function (taken approximately (as well as if the gradients are gradient) for constrained convex optimization). For such extreme point of the domain, the algorithm considers a low-rank update, and discusses the usefulness even for other optimizers. This result is obtained by extending the duality concept as well as the algorithm itself are fully invariant under any affine transformation/pre-conditioning. This approach is useful even for other optimizers. This result is obtained by extending the duality concept as well as the algorithm itself are fully invariant under any affine transformation/pre-conditioning.

(From Jaggi 2011)
Norm constraints

What happens when $C = \{x : \|x\| \leq t\}$ for a norm $\| \cdot \|$? Then

$$s \in \arg\min_{\|s\| \leq t} \nabla f(x^{(k-1)})^T s$$

$$= -t \cdot \left( \arg\max_{\|s\| \leq 1} \nabla f(x^{(k-1)})^T s \right)$$

$$= -t \cdot \partial \| \nabla f(x^{(k-1)}) \|_*$$

where $\| \cdot \|_*$ denotes the corresponding dual norm. That is, if we know how to compute subgradients of the dual norm, then we can easily perform Frank-Wolfe steps.

A key to Frank-Wolfe: this can often be simpler or cheaper than projection onto $C = \{x : \|x\| \leq t\}$.
Today:

- Examples
- Convergence analysis
- Properties and variants
- Path following
Example: $\ell_1$ regularization

For the $\ell_1$-regularized problem

$$\min_x f(x) \text{ subject to } \|x\|_1 \leq t$$

we have $s^{(k-1)} \in -t\partial\|\nabla f(x^{(k-1)})\|_\infty$. Frank-Wolfe update is thus

$$i_{k-1} \in \arg\max_{i=1,\ldots,p} |\nabla_i f(x^{(k-1)})|$$

$$x^{(k)} = (1 - \gamma_k)x^{(k-1)} - \gamma_k t \cdot \text{sign}(\nabla_{i_{k-1}} f(x^{(k-1)})) \cdot e_{i_{k-1}}$$

Like greedy coordinate descent! (But with diminishing steps)

Note: this is a lot simpler than projection onto the $\ell_1$ ball, though both require $O(n)$ operations
Example: $\ell_p$ regularization

For the $\ell_p$-regularized problem

$$\min_x f(x) \text{ subject to } \|x\|_p \leq t$$

for $1 \leq p \leq \infty$, we have $s^{(k-1)} \in -t\partial\|\nabla f(x^{(k-1)})\|_q$, where $p, q$ are dual, i.e., $1/p + 1/q = 1$. Claim: can choose

$$s_i^{(k-1)} = -\alpha \cdot \text{sign} (\nabla f_i(x^{(k-1)})) \cdot \|\nabla f_i(x^{(k-1)})\|^{p/q}, \quad i = 1, \ldots, n$$

where $\alpha$ is a constant such that $\|s^{(k-1)}\|_q = t$ (check this!), and then Frank-Wolfe updates are as usual.

Note: this is a lot simpler projection onto the $\ell_p$ ball, for general $p$! Aside from special cases ($p = 1, 2, \infty$), these projections cannot be directly computed (must be treated as an optimization)
Example: trace norm regularization

For the trace-regularized problem

$$\min_X f(X) \text{ subject to } \|X\|_{tr} \leq t$$

we have $S^{(k-1)} \in -t\partial\|\nabla f(X^{(k-1)})\|_{op}$. Claim: can choose

$$S^{(k-1)} = -t \cdot uv^T$$

where $u, v$ are leading left and right singular vectors of $\nabla f(X^{(k-1)})$ (check this!), and then Frank-Wolfe updates are as usual

Note: this substantially simpler and cheaper than projection onto the trace norm ball, which requires a singular value decomposition!
Constrained and Lagrange forms

Recall that solution of the constrained problem

$$\min_x f(x) \text{ subject to } \|x\| \leq t$$

are equivalent to those of the Lagrange problem

$$\min_x f(x) + \lambda \|x\|$$

as we let the tuning parameters $t$ and $\lambda$ vary over $[0, \infty]$. Typically in statistics and ML problems, we would just solve whichever form is easiest, over wide range of parameter values, then use CV.

So we should also compare the Frank-Wolfe updates under $\| \cdot \|$ to the proximal operator of $\| \cdot \|$.
• \(\ell_1\) norm: Frank-Wolfe update scans for maximum of gradient; proximal operator soft-thresholds the gradient step; both use \(O(n)\) flops

• \(\ell_p\) norm: Frank-Wolfe update computes raises each entry of gradient to power and sums, in \(O(n)\) flops; proximal operator not generally directly computable

• Trace norm: Frank-Wolfe update computes top left and right singular vectors of gradient; proximal operator soft-thresholds the gradient step, requiring a singular value decomposition

Various other constraints yield efficient Frank-Wolfe updates, e.g., special polyhedra or cone constraints, sum-of-norms (group-based) regularization, atomic norms. See Jaggi (2011)
Example: lasso comparison

Comparing projected and conditional gradient for constrained lasso problem, with $n = 100$, $p = 500$:

Note: FW uses standard step sizes, line search would probably help
Duality gap

Frank-Wolfe iterations admit a very natural duality gap:

\[ g(x^{(k)}) = \nabla f(x^{(k)})^T (x^{(k)} - s^{(k)}) \]

Claim: it holds that \( f(x^{(k)}) - f^* \leq g(x^{(k)}) \)

Proof: by the first-order condition for convexity

\[ f(s) \geq f(x^{(k)}) + \nabla f(x^{(k)})^T (s - x^{(k)}) \]

Minimizing both sides over all \( s \in C \) yields

\[ f^* \geq f(x^{(k)}) + \min_{s \in C} \nabla f(x^{(k)})^T (s - x^{(k)}) \]

\[ = f(x^{(k)}) + \nabla f(x^{(k)})^T (s^{(k)} - x^{(k)}) \]

Rearranged, this gives the duality gap above
Why do we call it “duality gap”? Rewrite original problem as

$$\min_x f(x) + I_C(x)$$

where $I_C$ is the indicator function of $C$. The dual problem is

$$\max_u -f^*(u) - I_C^*(-u)$$

where $I_C^*$ is the support function of $C$. Duality gap at $x, u$ is

$$f(x) + f^*(u) + I_C^*(-u) \geq x^T u + I_C^*(-u)$$

Evaluated at $x = x^{(k)}$, $u = \nabla f(x^{(k)})$, this gives

$$\nabla f(x^{(k)})^T x^{(k)} + \max_{s \in C} -\nabla f(x^{(k)})^T s = \nabla f(x^{(k)})^T (x^{(k)} - s^{(k)})$$

which is our gap
Convergence analysis

Following Jaggi (2011), define the curvature constant of $f$ over $C$:

$$M = \max_{\gamma \in [0,1]} \frac{2}{\gamma^2} \left( f(y) - f(x) - \nabla f(x)^T (y - x) \right)$$

$$y = (1 - \gamma)x + \gamma s$$

Note that $M = 0$ for linear $f$, and $f(y) - f(x) - \nabla f(x)^T (y - x)$ is called the Bregman divergence, defined by $f$

**Theorem:** The Frank-Wolfe method using standard step sizes $\gamma_k = 2/(k + 1)$, $k = 1, 2, 3, \ldots$ satisfies

$$f(x^{(k)}) - f^* \leq \frac{2M}{k + 2}$$

Thus number of iterations needed for $f(x^{(k)}) - f^* \leq \epsilon$ is $O(1/\epsilon)$
This matches the sublinear rate for projected gradient descent for Lipschitz $\nabla f$, but how do the assumptions compare?

For Lipschitz $\nabla f$ with constant $L$, recall

$$f(y) - f(x) - \nabla f(x)^T (y - x) \leq \frac{L}{2} \|y - x\|_2^2$$

Maximizing over all $y = (1 - \gamma)x + \gamma s$, and multiplying by $2/\gamma^2$,

$$M = \max_{\gamma \in [0,1], x,s,y \in C, y=(1-\gamma)x+\gamma s} \frac{2}{\gamma^2} \cdot \frac{L}{2} \|y - x\|_2^2$$

$$M = \max_{x,s \in C} L \|x - s\|_2^2 = L \cdot \text{diam}^2(C)$$

Hence assuming a bounded curvature is basically no stronger than what we assumed for projected gradient
Basic inequality

The key inequality used to prove the Frank-Wolfe convergence rate:

\[ f(x^{(k)}) \leq f(x^{(k-1)}) - \gamma_k g(x^{(k-1)}) + \frac{\gamma_k^2}{2} M \]

Here \( g(x) = \max_{s \in C} \nabla f(x)^T (x - s) \) is duality gap defined earlier

Proof: write \( x^+ = x^{(k)} \), \( x = x^{(k-1)} \), \( s = s^{(k-1)} \), \( \gamma = \gamma_k \). Then

\[ f(x^+) = f(x + \gamma (s - x)) \]
\[ \leq f(x) + \gamma \nabla f(x)^T (s - x) + \frac{\gamma^2}{2} M \]
\[ = f(x) - \gamma g(x) + \frac{\gamma^2}{2} M \]

Second line used definition of \( M \), and third line the definition of \( g \)
The proof of the convergence result is now straightforward. Denote by \( h(x) = f(x) - f^* \) the suboptimality gap at \( x \). Basic inequality:

\[
h(x^{(k)}) \leq h(x^{(k-1)}) - \gamma_k g(x^{(k-1)}) + \frac{\gamma_k^2}{2} M
\]

\[
\leq h(x^{(k-1)}) - \gamma_k h(x^{(k-1)}) + \frac{\gamma_k^2}{2} M
\]

\[
= (1 - \gamma_k) h(x^{(k-1)}) + \frac{\gamma_k^2}{2} M
\]

where in the second line we used \( g(x^{(k-1)}) \geq h(x^{(k-1)}) \)

To get the desired result we use induction:

\[
h(x^{(k)}) \leq \left( 1 - \frac{2}{k+1} \right) \frac{2M}{k+1} + \left( \frac{2}{k+1} \right)^2 \frac{M}{2} \leq \frac{2M}{k+2}
\]
Affine invariance

Frank-Wolfe updates are **affine invariant**: for nonsingular matrix $A$, define $x = Ax'$, $F(x') = f(Ax')$, consider Frank-Wolfe on $F$:

$$s' = \arg\min_{z \in A^{-1}C} \nabla F(x')^T z$$

$$(x')^+ = (1 - \gamma)x' + \gamma s'$$

Multiplying by $A$ produces same Frank-Wolfe update as that from $f$. Convergence analysis is also affine invariant: curvature constant

$$M = \max_{\gamma \in [0,1]} \frac{2}{\gamma^2} \left( F(y') - F(x') - \nabla F(x')^T (y' - x') \right)$$

matches that of $f$, because $\nabla F(x')^T (y' - x') = \nabla f(x)^T (y - x)$
Inexact updates

Jaggi (2011) also analyzes inexact Frank-Wolfe updates: suppose we choose $s^{(k-1)}$ so that

$$\nabla f(x^{(k-1)})^T s^{(k-1)} \leq \min_{s \in C} \nabla f(x^{(k-1)})^T s + \frac{M \gamma_k}{2} \cdot \delta$$

where $\delta \geq 0$ is our inaccuracy parameter. Then we basically attain the same rate

**Theorem:** Frank-Wolfe using step sizes $\gamma_k = 2/(k + 1)$, $k = 1, 2, 3, \ldots$, and inaccuracy parameter $\delta \geq 0$, satisfies

$$f(x^{(k)}) - f^* \leq \frac{2M}{k + 1} (1 + \delta)$$

Note: the optimization error at step $k$ is $M \gamma_k / 2 \cdot \delta$. Since $\gamma_k \to 0$, we require the errors to vanish
Two variants

Two important variants of Frank-Wolfe:

- **Line search**: instead of using standard step sizes, use

  \[
  \gamma_k = \arg\min_{\gamma \in [0,1]} f \left( x^{(k-1)} + \gamma(s^{(k-1)} - x^{(k-1)}) \right)
  \]

  at each \( k = 1, 2, 3, \ldots \). Or, we could use backtracking

- **Fully corrective**: directly update according to

  \[
  x^{(k)} = \arg\min_y f(y) \quad \text{subject to} \quad y \in \text{conv}\{x^{(0)}, s^{(0)}, \ldots, s^{(k-1)}\}
  \]

Both variants lead to the same \( O(1/\epsilon) \) iteration complexity

Another popular variant: **away steps**, which get linear convergence under strong convexity
Path following

Given the norm constrained problem

$$\min_x f(x) \text{ subject to } \|x\| \leq t$$

Frank-Wolfe can be used for path following, i.e., we can produce an approximate solution path $\hat{x}(t)$ that is $\epsilon$-suboptimal for every $t \geq 0$. Let $t_0 = 0$ and $x^*(0) = 0$, fix $m > 0$, repeat for $k = 1, 2, 3, \ldots$:

- Calculate
  $$t_k = t_{k-1} + \frac{(1 - 1/m)\epsilon}{\|\nabla f(\hat{x}(t_{k-1}))\|_*}$$
  and set $\hat{x}(t) = \hat{x}(t_{k-1})$ for all $t \in (t_{k-1}, t_k)$

- Compute $\hat{x}(t_k)$ by running Frank-Wolfe at $t = t_k$, terminating when the duality gap is $\leq \epsilon/m$

(This is a simplification of the strategy from Giesen et al., 2012)
Claim: this produces (piecewise-constant) path with
\[ f(\hat{x}(t)) - f(x^*(t)) \leq \epsilon \quad \text{for all } t \geq 0 \]

Proof: rewrite the Frank-Wolfe duality gap as
\[
    g_t(x) = \max_{\|s\| \leq t} \nabla f(x)^T(x - s) = \nabla f(x)^T x + t\|\nabla f(x)\|_*
\]

This is a linear function of \( t \). Hence if \( g_t(x) \leq \epsilon/m \), then we can increase \( t \) until \( t^+ = t + (1 - 1/m)\epsilon/\|\nabla f(x)\|_* \), because

\[
    g_{t^+}(x) = \nabla f(x)^T x + t\|\nabla f(x)\|_* + \epsilon - \epsilon/m \leq \epsilon
\]
i.e., the duality gap remains \( \leq \epsilon \) for the same \( x \), between \( t \) and \( t^+ \)
References