$$g(u,v) \le f^* \quad \text{all } \overset{\text{dual}}{\text{feasible}}\ u,v.$$

$$\Longrightarrow \quad f(x) - f^* \le f(x) - g(u,v)$$

and $\quad g^* - g(u,v) \le f(x) - g(u,v)$

$$\min_x \sum_{i=1}^{n} \left(f_i(x_i) - v a_i x_i \right)$$

$$= \sum_{i=1}^{n} \underbrace{\min_{x_i} \left(f_i(x_i) - v a_i x_i \right)}_{\overset{\text{def}}{=} -f_i^*(v a_i)}.$$

---

x_i^* solves $\displaystyle \min_{x_i} \left(f_i(x_i) - v^* a_i x_i \right)$

$y = \dfrac{z}{\|z\|}$. observe $\|y\| \le 1$

$$|y^T x| \le \max_{\|w\| \le 1} w^T x = \|x\|_*$$

thus $\quad |z^T x| \le \|z\|\, \|x\|_*$

dual pairs

- $p=2,\ q=2$
- $p=1,\ q=\infty$.

$$f^{\star} \qquad f^{*}$$

$$f^*(y) = \max_z \; y^T z - f(z)$$

$$\geq y^T x - f(x)$$

$$\Rightarrow f(x) + f^*(y) \geq y^T x.$$

---

f is called closed if all sublevel sets are closed

i.e. $\{x : f(x) \leq t\}$ closed for all t

---

when f, f^* are differentiable?

$$x = \nabla f^*(y) \iff y = \nabla f(x).$$

i.e. $(\nabla f)^{-1} = \nabla f^*$

$$f(x) = \frac{1}{2} x^T Q x \qquad Q \succ 0.$$

$$f^*(y) = \max_x \; \underbrace{y^T x - \frac{1}{2} x^T Q x.}$$

$$0 = y - Qx.$$

$$x = Q^{-1} y.$$

$$f^*(y) = y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} y$$

$$= \frac{1}{2} y^T Q^{-1} y.$$

Fenchel: $\frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y \geq x^T y$

$f(x) = I_C(x)$ $\quad$ C closed and convex.

$$f^*(y) = \max \; y^T x - I_C(x)$$
$$= \max_{x \in C} \; y^T x.$$
$$= I_C^*(y)$$ support function of C.

$$I_C^{**}(x) = I_C(x)$$

implication of this rule:

$$\|x\| = \|x\|_{**} = \max_{\underbrace{\|z\|_* \le 1}} \; z^T x.$$

~~this is my set C~~

$$\left(\max_{\|z\|_* \le 1} \; z^T x \right)^* = I_{\{\|z\|_* \le 1\}}(z).$$

Lasso: $\quad g(\beta, \emptyset) = \min_\beta \; f(\beta) = f^*.$

$$g(u) = \min_{\beta, z} \; \tfrac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$$
$$= \underbrace{\min_z \; \tfrac{1}{2}\|y - z\|_2^2 + u^T z} + \underbrace{\min_\beta \; \lambda\|\beta\|_1 - u^T X\beta}$$

$0 = z - y + u$

$\underline{z = y - u.}$

$(**)$

$$= \tfrac{1}{2}\|u\|_2^2 + u^T(y - u)$$
$$= \tfrac{1}{2}\|y\|_2^2 - \tfrac{1}{2}\|y - u\|_2^2$$

$$-\lambda \max_\beta \left(\frac{u^T X}{\lambda}\beta - \|\beta\|_1 \right)$$

$(*)$

(*) This is $\| \|_1^*$ evaluated at $\frac{X^T u}{\lambda}$

$$= I_{\{\|z\|_\infty \le 1\}}\left(\frac{X^T u}{\lambda}\right)$$

(**) : recall that $z = y - u$.

But $z = X\beta$, so $X\beta = y - u$

i.e. $u = y - X\beta$

Dual is : $\min_{\text{s.t.}} \|y-u\|_2^2$ s.t. $\|X^T u\|_\infty \le \lambda$

project y onto $\underbrace{\{u : \|X^T u\|_\infty \le \lambda\}}_{\text{polyhedron. } C}$