Supplement to: A Statistician Plays Darts

Rearranging the Dartboard

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Recall that we considered the simple model for dart throws

$$Z = \mu + \epsilon,\ \epsilon \sim \mathcal{N}(0, \sigma^2 I),$$

and computed $\mathrm{E}_{\mu,\sigma^2}[s(Z)]$ for a given σ and all μ over a fine grid. We were concerned mainly with the optimal location

$$\underset{\mu}{\operatorname{argmax}}\ \mathrm{E}_{\mu,\sigma^2}[s(Z)],$$

and we noted that this varies considerably with σ. Now we turn our attention to optimal expected score

$$f(\sigma) = \max_{\mu}\ \mathrm{E}_{\mu,\sigma^2}[s(Z)].$$

Not surprisingly, this drops significantly with increasing σ, shown in Figure 1. For $0 \leq \sigma \leq 20$, this curve behaves like $2^{-\sigma}$, and then it decreases linearly for $20 < \sigma \leq 100$. Thus for a skilled player ($\sigma \leq 20$) every increase in accuracy reaps large rewards. On the other hand, it appears than an unskilled player ($\sigma \geq 60$) can't do much better than the uniform model!

The sharp decline over $0 \leq \sigma \leq 20$ can be regarded as a testament to the difficulty of the current dartboard. This raises the question: can we rearrange the numbers $1, \ldots 20$ to produce an even harder dartboard (sharper decline)? We measure the difficulty of a dartboard arrangement by

$$\int_{15}^{60} f_d(\sigma)\ d\sigma,$$

where

$$f_d(\sigma) = \max_{\mu}\ \mathrm{E}_{\mu,\sigma^2}[s_d(Z)],$$

with s_d the score function for dartboard arrangement $d = \big(d(1), \ldots d(20)\big)$. Figure

We first consider two alternate arrangements. The first is

$$d_{\text{Curtis}} = (20, 1, 19, 3, 17, 5, 15, 7, 13, 9, 11, 10, 12, 8, 14, 6, 16, 4, 18, 2),$$

taken from [Cur04]. This arrangement maximizes the sum of the absolute adjacent differences $P_1(d) = \sum_{i=1}^{20} |d(i+1) - d(i)|$, where we let $d(21) = d(1)$. The second is

$$d_{\text{linear}} = (20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1),$$

a simple linear arrangement. Intuitively, we expect that the arrangement d_{Curtis} will be quite hard, but d_{linear} should be pretty easy. Figure 2 visualizes the different dartboard arrangements.

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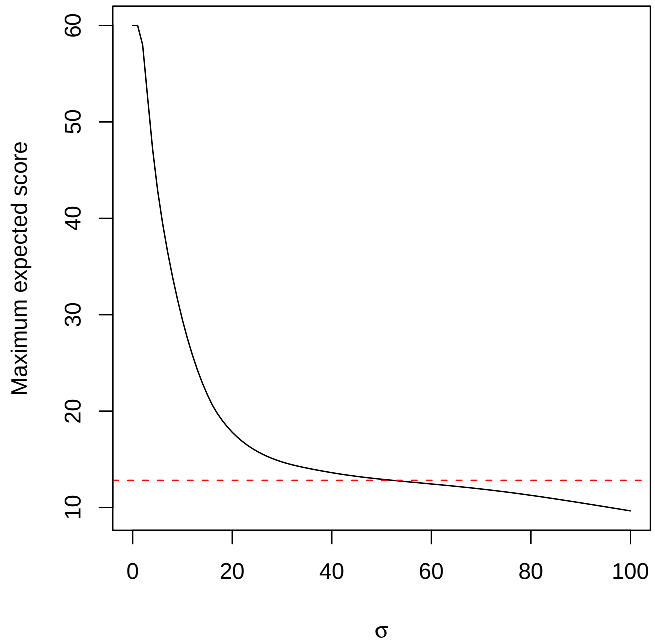

Figure 1: *Plot of the maximum expected score* $f(\sigma) = \max_\mu \mathrm{E}_{\mu,\sigma^2}[s(Z)]$ *over the range* $0 \leq \sigma \leq 100$. *The red dashed line corresponds to the average score when the dart throw is distributed uniformly at random over the board.*

We also consider a search over all possible dartboard arrangements based on the Metropolis-Hastings algorithm (c.f [Liu08] for a complete description of Metropolis-Hastings and other Markov Chain Monte Carlo techniques) to sample a random dartboard D according to

$$\mathrm{P}_\theta(D = d) \propto \exp\left(-\theta \int_{15}^{60} f_d(\sigma)\, d\sigma\right). \tag{1}$$

The interval $[15, 60]$ was chosen as nearly all dartboard arrangements seem to agree for $\sigma < 15$, and the challenging ones agree for $\sigma > 60$.

Our algorithm can be described in two simple steps, following the general Metropolis-Hastings steps:

Proposal: Given a current arrangement $D_t = d$ at time t, generate a new arrangement $d_{\{i,j\}}$ by swapping the position of two elements of the arrangement, chosen uniformly at random.

Acceptance: Simulate $U \sim \mathrm{Uniform}(0, 1)$, if $U \leq \mathrm{P}_\theta(d_{\{i,j\}})/\mathrm{P}_\theta(d)$ then accept the proposal (i.e. set $D_{t+1} = d_{\{i,j\}}$), else remain at d (i.e. set $D_{t+1} = d$).

This algorithm constructs a random walk over dartboard arrangements whose stationary distribution (1) gives higher probability to boards with consistently small values of f_d. In order to find the most difficult arrangement, the simplest approach is to run the algorithm for T time steps, yielding a sequence of arrangements $(D_1, \ldots, D_T)$, returning

$$D^* = \underset{d \in \{D_1, \ldots, D_T\}}{\mathrm{argmin}} \int_{15}^{60} f_d(\sigma)\, d\sigma.$$

We chose this naive method for finding the arrangement with lowest score over more sophisticated techniques such as stochastic annealing [GG84].

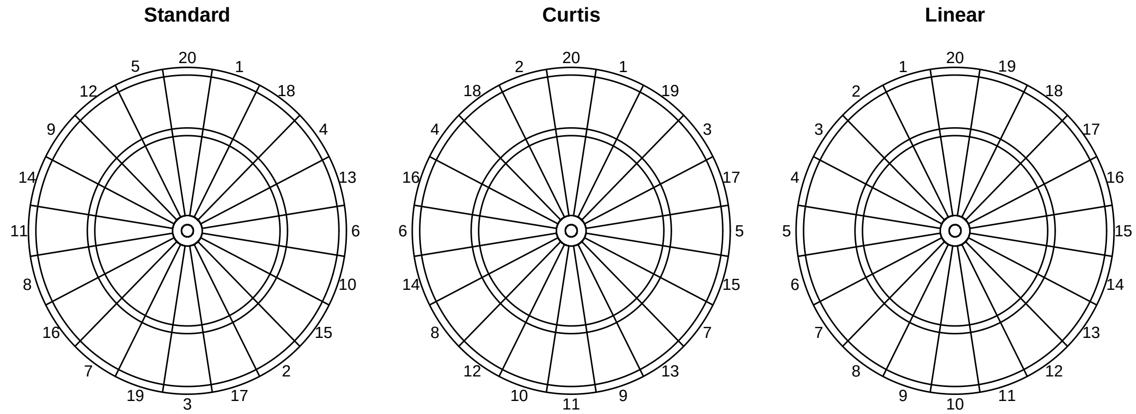

Figure 2:

See Figure 3 for a plot of f_d for the various arrangements d. Over the interval $[15, 60]$, it turns out that $f_{d_{\text{Curtis}}} < f_{d_{\text{standard}}}$, while $f_{d_{\text{linear}}} \gg f_{d_{\text{standard}}}$. Starting at the Curtis arrangement, we ran the Metropolis-Hastings algorithm for many time steps. Interestingly, the best arrangement that we encountered, D^*, is actually just a reflection of the Curtis board about the y-axis. We note that D^* has the same absolute adjacent differences as the Curtis arrangement, so it is also maximal with respect to P_1. The curves f_{D^*} and $f_{d_{\text{Curtis}}}$ are equal (up to small numerical errors) for every value of σ, as it should be, given the symmetry of our Gaussian distribution.

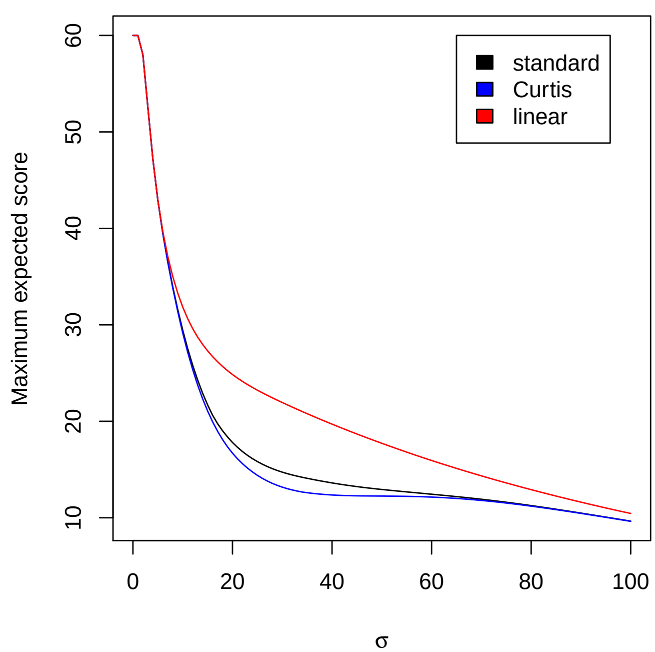

Figure 3: *Plot of the maximum expected score f_d for the various dartboard arrangements d.*

Furthermore, for every t, the visited chain D_t achieved

$$f_{D_t}(\sigma) \geq f_{d_{\text{Curtis}}}(\sigma), \quad 15 \leq \sigma \leq 60.$$

This leads us to the following conjecture (which we will not attempt to prove)

$$d_{\mathrm{Curtis}} = \operatorname*{argmin}_{d} f_d(\sigma), \quad 15 \leq \sigma \leq 60.$$

References

[Cur04] S. A. Curtis. Darts and hoopla board design. *Information Processing Letters*, 92(1):53–56, 2004.

[GG84] S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE transactions on pattern analysis and machine intelligence*, 6(6):721–741, 1984.

[Liu08] J. S. Liu. *Monte Carlo strategies in scientific computing.* Springer Series in Statistics. Springer, New York, 2008.