Kalman Filter, Sensor Fusion, and Constrained Regression: Equivalences and Insights

David C. Farrow  Maria Jahja  Roni Rosenfeld  Ryan J. Tibshirani

Abstract

The Kalman filter (KF) is one of the most widely used tools for data assimilation and sequential estimation. In this paper, we show that the state estimates from the KF in a standard linear dynamical system setting are exactly equivalent to those given by the KF in a transformed system, with infinite process noise (a “flat prior”) and an augmented measurement space. This reformulation—which we refer to as augmented measurement sensor fusion (SF)—is conceptually interesting, because the transformed system here is seemingly static (as there is effectively no process model), but we can still capture the state dynamics inherent to the KF by folding the process model into the measurement space. Apart from being interesting, this reformulation of the KF turns out to be useful in problem settings in which past states are eventually observed (at some lag). In such problems, when we use the empirical covariance to estimate the measurement noise covariance, we show that the state predictions from augmented measurement SF are exactly equivalent to those from a regression of past states on past measurements, subject to particular linear constraints (reflecting the relationships encoded in the measurement map). This allows us to port standard ideas (say, regularization methods) in regression over to dynamical systems. For example, we can posit multiple candidate process models, fold all of them into the measurement model, transform to the regression perspective, and apply $\ell_1$ penalization to perform process model selection. We give various empirical demonstrations, and focus on an application to nowcasting the weekly incidence of influenza in the US.

1 Introduction

Let $x_t \in \mathbb{R}^k, t = 1, 2, 3, \ldots$ denote states and $z_t \in \mathbb{R}^d, t = 1, 2, 3, \ldots$ denote measurements evolving according to the time-invariant linear dynamical system:

$$x_t = F x_{t-1} + \delta_t,$$

$$z_t = H x_t + \epsilon_t,$$

for $t = 1, 2, 3, \ldots$. We assume the noise terms $\delta_t, \epsilon_t$ have mean zero and covariances $Q \in \mathbb{R}^{k \times k}$ and $R \in \mathbb{R}^{d \times d}$, respectively, for all $t = 1, 2, 3, \ldots$. Also, we assume that the initial state $x_0$ and all noise terms are mutually independent. We call (1) the process model and (2) the measurement model.

Kalman filter. The Kalman filter (KF) [Kalman 1960] is a method for sequential estimation in the model (1), (2). Given past estimates $\hat{x}_1, \ldots, \hat{x}_t$ and measurements $z_1, \ldots, z_{t+1}$, we form an estimate $\hat{x}_{t+1}$ of the state $x_{t+1}$ via

$$\hat{x}_{t+1} = F \hat{x}_t,$$

$$\hat{x}_{t+1} = \hat{x}_{t+1} + K_{t+1} (z_{t+1} - H \hat{x}_{t+1}),$$

where $K_{t+1}$ is the Kalman gain, which is determined by the covariance matrices $Q$ and $R$. This gain reflects the relative importance of the process and measurement models in updating the state estimate.
where $K_{t+1} \in \mathbb{R}^{k \times d}$ is called the Kalman gain (at time $t + 1$). It is itself updated sequentially, via

$$
P_{t+1} = FP_{t}F^{T} + Q, \quad (5)$$

$$
K_{t+1} = P_{t+1}H^{T}(HP_{t+1}H^{T} + R)^{-1}, \quad (6)$$

$$
P_{t+1} = (I - K_{t+1}H)P_{t+1}. \quad (7)$$

where $P_{t+1} \in \mathbb{R}^{k \times k}$ denotes the state error covariance (at time $t + 1$). The step (5) is often called the predict step: we form an intermediate estimate $\hat{x}_{t+1}$ of the state based on the process model and our estimate at the previous time point. The step (4) is often called the update step: we update our estimate $\hat{x}_{t+1}$ based on the measurement model and the measurement $z_{t+1}$.

Under the data model (1), (2) and the conditions on the noise stated above, the Kalman filter attains the optimal mean squared error $\mathbb{E}\|\hat{x}_{t} - x_{t}\|_{2}^{2}$ among all linear unbiased filters, at each $t = 1, 2, 3, \ldots$. When the initial state $x_{0}$ and all noise terms are Gaussian, the Kalman filter estimates exactly reduce to the Bayes estimates $\hat{x}_{t} = \mathbb{E}(x_{t}|z_{1}, \ldots, z_{t}), t = 1, 2, 3, \ldots$. Numerous important extensions have been proposed, e.g., the ensemble Kalman filter (EnKF) [Evensen 1994, Houtekamer and Mitchell 1998], which approximates the noise process covariance $Q$ by a sample covariance in an ensemble of state predictions, as well as the extended Kalman filter (EKF) [Smith et al. 1962] and unscented Kalman filter (UKF) [Julier and Uhlmann 1997], which both allow for nonlinearities in the process model. Particle filtering (PF) [Gordon et al. 1993] has recently become a very popular approach for modeling complex dynamics. PF adaptively approximates the posterior distribution, and in doing so, avoids the linear, Gaussian assumptions inherent to the KF. This flexibility comes at the cost of a greater computational burden.

In this paper, we revisit the standard KF (3), (4) and show that its estimates $\hat{x}_{t+1}, t = 0, 1, 2, \ldots$ are equivalent to those from the KF applied to a transformed system, with infinite process noise and an augmented measurement space. At first glance, this is perhaps surprising, because the transformed system effectively lacks a process model and is therefore seemingly static; however, it is able to take the state dynamics into account as part of its measurement model. Importantly, this reformulation of the KF leads us to derive a second, key reformulation for problems in which past states are observed (at some lag). This second reformulation is the methodological crux of our paper: it is a constrained regression approach for predicting states from measurements, motivated by (derived from) SF and the KF. We illustrate its effectiveness in an application to nowcasting weekly influenza levels in the US.

Sensor fusion. If we let the noise covariance in the process model diverge to infinity, $Q \rightarrow \infty$, then the Kalman filter estimate in (3), (4) simplifies to

$$
\hat{x}_{t+1} = (H^{T}R^{-1}H)^{-1}H^{T}R^{-1}z_{t+1}. \quad (8)
$$

This can be verified by rewriting the Kalman gain as $K_{t+1} = (P_{t+1}^{-1} + H^{T}R^{-1}H)^{-1}H^{T}R^{-1}$, and observing that $P_{t+1}^{-1} \rightarrow 0$ as $Q \rightarrow \infty$. Alternatively, we can verify this by specializing to the case of Gaussian noise: as $\text{tr}(Q) \rightarrow \infty$, we approach a flat prior, and the Kalman filter (Bayes estimator) just maximizes the likelihood of $z_{t+1}|x_{t+1}$. From the measurement model (2) (assuming Gaussian noise), this is a weighted regression of $z_{t+1}$ on the measurement map $H$, precisely as in (8).

We will call (8) the sensor fusion (SF) estimate (at time $t + 1$). In this setting, we will also refer to the measurements as sensors. As defined, sensor fusion is a special case of the Kalman filter when there is infinite process noise; said differently, it is a special case of the Kalman filter when there is no a process model at all. Thus, looking at (8), the state dynamics have apparently been completely lost. Somewhat surprisingly, as we will show shortly, they can be exactly recovered by augmenting the measurement vector $z_{t+1}$ with the KF intermediate prediction $\bar{x}_{t+1}$ in (3) (and adjusting the map $H$ and covariance $R$ appropriately). We summarize this and our other contributions next.

Summary of contributions. An outline of our contributions in this paper is as follows.

1. We show in Section 2 that, if we take the KF intermediate prediction $\bar{x}_{t+1}$ in (3), append it to the measurement vector $z_{t+1}$, and perform SF (with an appropriately adjusted $H, R$), then the result is exactly the KF estimate (4).

To make this unambiguous, we may take, say, $Q = aI$ and let $a \rightarrow \infty$.

2. "Sensor fusion" is typically used as a generic term, similar to "data assimilation"; we use it to specifically describe the estimate in (8) to distinguish it from the KF. This is useful when we describe equivalences, shortly.
2. We show in Section 3 that, if we are in a problem setting in which past states are observed (at some lag, which is the case in the flu nowcasting application), and we replace the noise covariance $R$ from the measurement model by the empirical covariance on past data, then the sensor fusion estimate (8) can be written as $B^T z_{t+1}$, where $B \in \mathbb{R}^{d \times k}$ is a matrix of coefficients that solves a regression problem of the states on the measurements (using past data), subject to the equality constraint $H^T B = I$.

3. We demonstrate the effectiveness of our new regression formulation of SF in Section 4 by describing an application of this methodology to nowcasting the incidence of weekly flu in the US. This achieves state-of-the-art performance in this problem.

4. We give in Section 5 some extensions of the regression formulation of SF; they do not have direct equivalences to SF (or the KF), but are intuitive extend dynamical systems modeling in new directions (e.g., using $\ell_1$ penalization to perform a kind of process model selection).

We make several remarks. The equivalences described in points 1–3 above are deterministic (they do not require the modeling assumptions (1), (2), or any modeling assumptions whatsoever). Further, even though their proofs are elementary (they are purely linear algebraic) and the setting is a classical one (linear dynamical systems), these equivalences are—as far as we can tell—new results. They deserve to be widely known and could have implications beyond what is explored in this paper.

For example, the regression formulation of SF may still be a useful perspective for problems in which past states are fully unobserved (this being the case in most KF applications). In such problems, we may consider using smoothed estimates of past states, obtained by running a backward version of the KF forward recursions (3)–(7) (see, e.g., Chapter 7 of Anderson and Moore (1979)), for the purposes of the regression formulation. As another example, the SF view of the KF may be a useful formulation for the purposes of estimating the covariances $F, H, R, Q$ (or the maps $F, H, R, Q$), in this paper, we assume that $F, H, R, Q$ are known (except for in the regression formulation of SF, in which $R$ is unknown but past states are available); in general, there are well-developed methods for estimating $F, H, R, Q$ such as subspace identification algorithms (see, e.g., Overshee and Moor (1996)), and it may be interesting to see if the SF perspective offers any advantages here.

Related work. The Kalman filter and its extensions, as previously referenced (EnKF, EKF, UKF), are the de facto standard in state estimation and tracking problems; the literature surrounding them is enormous and we cannot give a thorough treatment. Various authors have pointed out the simple fact that maximum likelihood estimate in (8), which we call sensor fusion, is the limit of the KF as the noise covariance in the process model approaches infinity (see, e.g., Chapter 5.9 of Brown and Hwang (2012)). We have not, however, seen any authors note that this static model can recover the KF by augmenting the measurement vector with the KF intermediate prediction (Theorem 1).

Along the lines of our second equivalence (Theorem 2), there is older work in the statistical calibration literature that studies the relationships between the regressions of $y$ on $x$ and $x$ on $y$ (for multivariate $x, y$, see Brown (1982)). This is somewhat related to our result, since we show that a backwards or indirect approach, which models $z_{t+1} \vert x_{t+1}$, is actually equivalent to a forwards or direct approach, which predicts $x_{t+1}$ from $z_{t+1}$ via regression. However, the details are quite different.

Finally, our SF methodology in the flu nowcasting application blends together individual predictors in a way that resembles linear stacking (Wolpert (1992) Breiman (1996). In fact, one implication of our choice of measurement map $H$ in the flu nowcasting problem, as well as the constraints in our regression formulation of SF, is that all regression weights must sum to 1, which is the standard in linear stacking as well. However, the equality constraints in our regression formulation are actually quite a bit more complex, and reflect aspects of the sensor hierarchy that linear stacking would not.

2 Equivalence between KF and SF

As already discussed, the sensor fusion estimate (8) is a limiting case of the Kalman filter (3)–(4), and initially, it seems, one rather limited in scope: there is effectively no process model (since we have sent the process variance to infinity). However, as we show next, the KF is actually itself a special case of SF, when we augment the measurement vector by the KF intermediate predictions, and appropriately adjust the measurement map $H$ and noise covariance $R$. The proof is elementary, a consequence of the Woodbury matrix and related manipulations. It is given in the appendix.
Theorem 1. At each time \( t = 0, 1, 2, \ldots \), suppose we augment our measurement vector by defining \( \tilde{z}_{t+1} = (z_{t+1}, \tilde{x}_{t+1}) \in \mathbb{R}^{d+k} \), where \( \tilde{x}_{t+1} = F\tilde{x}_t \) is the KF intermediate prediction at time \( t+1 \). Suppose that we also augment our measurement map by defining \( H \in \mathbb{R}^{(d+k) \times k} \) to be the rowwise concatenation of \( H \) and the identity matrix \( I \in \mathbb{R}^{k \times k} \). Furthermore, suppose we define an augmented measurement noise covariance
\[
\tilde{R}_{t+1} = \begin{bmatrix} R & 0 \\ 0 & \tilde{P}_{t+1} \end{bmatrix},
\tag{9}
\]
where \( \tilde{P}_{t+1} \) is the KF intermediate error covariance at time \( t+1 \) (as in (5)). Then applying SF to the augmented system produces an estimate at \( t+1 \) that equals the KF estimate,
\[
(\tilde{H}^T \tilde{R}_{t+1}^{-1} \tilde{H})^{-1} \tilde{H}^T \tilde{R}_{t+1}^{-1} \tilde{z}_{t+1} = \tilde{x}_{t+1} + \tilde{K}_{t+1}(\tilde{z}_{t+1} - \tilde{H}\tilde{x}_{t+1}),
\tag{10}
\]
where \( \tilde{K}_{t+1} \) is the Kalman gain at \( t+1 \) (as in (6)).

Remark 1. We can think of the last state estimate \( \tilde{x}_t \) in the theorem (which is propagated forward via \( \tilde{x}_{t+1} = F\tilde{x}_t \)) as the previous output from SF itself, when applied to the appropriate augmented system. More precisely, by induction, Theorem 1 says that iteratively applying SF to \( z_{t+1}, H, \tilde{R}_{t+1} \) across times \( t = 0, 1, 2, \ldots \), where each \( \tilde{x}_{t+1} = F\tilde{x}_t \) is the intermediate prediction using the last SF estimate \( \tilde{x}_t \), produces a sequence \( \tilde{x}_{t+1}, t = 0, 1, 2, \ldots \) that matches the state estimates from the KF.

Remark 2. The equivalence between SF and KF can be extended beyond the case of linear process and linear measurement models. Given a nonlinear process map \( f \) and a nonlinear process model \( h \), suppose we define \( \tilde{x}_{t+1} = f(\tilde{x}_t), F_{t+1} = Df(\tilde{x}_t) \) (the Jacobian of \( f \) at \( \tilde{x}_t \)), and \( H_{t+1} = Dh(\tilde{x}_{t+1}) \) (the Jacobian of \( h \) at \( \tilde{x}_{t+1} \)). Suppose we define the augmented measurement vector as
\[
\tilde{z}_{t+1} = (z_{t+1} + H_{t+1}\tilde{x}_{t+1} - h(\tilde{x}_{t+1}), \tilde{x}_{t+1}),
\tag{11}
\]
where we have offset the measurement \( z_{t+1} \) by the residual \( H_{t+1}\tilde{x}_{t+1} - h(\tilde{x}_{t+1}) \) from linearization. Suppose, as in the theorem, we define the augmented measurement map \( \tilde{H}_{t+1} \in \mathbb{R}^{(d+k) \times k} \) to be the rowwise concatenation of \( H_{t+1} \) and \( I \in \mathbb{R}^{k \times k} \), and define \( \tilde{R}_{t+1} \in \mathbb{R}^{(d+k) \times (d+k)} \) as in (9), for \( \tilde{P}_{t+1} \) the predicted error covariance as in (5), but with \( F_{t+1}, H_{t+1} \) in place of \( F, H \). In the appendix, we prove that
\[
(\tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{H}_{t+1})^{-1} \tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{z}_{t+1} = \tilde{x}_{t+1} + \tilde{K}_{t+1}(\tilde{z}_{t+1} - \tilde{H}\tilde{x}_{t+1}),
\tag{12}
\]
where \( \tilde{K}_{t+1} \) is as in (6), but with \( F_{t+1}, H_{t+1} \) in place of \( F, H \). The right-hand side above is precisely the extended KF (EK). The left-hand side is what we might call extended SF (ESF).

A simple empirical verification of Theorem 1 is given in the appendix.

3 Equivalence between SF and regression

Suppose that in our linear dynamical system, at each time \( t \), we observe the measurement \( z_t \), make a prediction \( \hat{x}_t \) for \( x_t \), then later observe the state \( x_t \) itself. (This setup indeed describes the influenza nowcasting problem, a central motivating example that we will describe shortly.) In such problems, we can estimate \( \hat{R} \) using the empirical covariance on past data. As we show below, when we plug this empirical covariance into (5), the SF estimate reduces to a prediction from a constrained regression of past states on past measurements. Next we give the details, then we describe the influenza nowcasting problem, a central applied point of motivation for us, which we use to interpret the constraints.

3.1 Equivalent regression problem

Recall, in making a state prediction at time \( t+1 \), we assume in this section that we have access to past states. We may assume without a loss of generality that we observe the full past \( x_i, i = 1, \ldots, t \) (if not, and we observe only some subset of the past, then the only changes to make in what follows are notational). Assuming the measurement noise covariance \( R \) is unknown, we may use
\[
\hat{R}_{t+1} = \frac{1}{t} \sum_{i=1}^{t} (z_i - Hx_i)(z_i - Hx_i)^T,
\tag{13}
\]
the empirical (uncentered) covariance based on past data, as an estimate. Under this choice, it turns out that sensor fusion (5) is exactly equivalent to a regression of states on measurements, subject to certain equality constraints. The proof is elementary, but requires detailed arguments. It is deferred until the appendix.
Theorem 2. Let $\hat{R}_{t+1}$ be as in (13) (assumed to be invertible). Consider the SF prediction at time $t+1$, with $R_{t+1}$ in place of $R$. Denote this by $\hat{x}_{t+1} = \hat{B}^T z_{t+1}$, where

$$\hat{B}^T = (H^T \hat{R}_{t+1}^{-1} H)^{-1} H^T \hat{R}_{t+1}^{-1}$$

(and $H^T \hat{R}_{t+1}^{-1} H$ is assumed invertible). Each column of $\hat{B}$, denoted $\hat{b}_j \in \mathbb{R}^d$, $j = 1, \ldots, k$, solves

$$\min_{b_j \in \mathbb{R}^d} \sum_{i=1}^t (x_{ij} - b_j^T z_i)^2$$

subject to $H^T b_j = e_j$, where $e_j \in \mathbb{R}^d$ is the $j$th standard basis vector (all 0s except for a 1 in the $j$th component).

Remark 3. As discussed in the introduction, the interpretation of $(H^T \hat{R}_{t+1}^{-1} H)^{-1} H^T \hat{R}_{t+1}^{-1} z_{t+1}$ as the coefficients from regressing $z_{t+1}$ (the response) onto $H$ (the covariates) is more or less immediate. Interpreting the same quantity as $B^T z_{t+1} = (b_1^T z_{t+1}, \ldots, b_k^T z_{t+1})$, the predictions from historically regressing $x_i$, $i = 1, \ldots, t$ (the response) onto $z_i$, $i = 1, \ldots, t$ (the covariates), however, is much less obvious. The latter is a forwards or direct regression approach to predicting $x_{t+1}$.

An empirical verification of Theorem 2 is given in Section 4. Before interpreting the constraints in (14), we discuss the flu nowcasting problem.

3.2 Influenza nowcasting

An example that we will revisit frequently, for the rest of the paper, is the following influenza (or flu) nowcasting problem. Here the state variable of interest is weekly percentage weighted influenza-like illness (wILI), a measure of flu incidence provided by the Centers for Disease Control and Prevention (CDC), in each of the $k = 51$ US states (including DC). Because it takes time for the CDC to collect and compile this data, they release wILI values with a 1 week delay. Meanwhile, various proxies for the flu (i.e., data sources that are potentially correlated with flu incidence) are available in real time, e.g., web search volume for flu-related terms, site traffic metrics for flu-related pages, pharmaceutical sales for flu-related products, etc. We can hence train (using historical data) sensors to predict wILI, one from each data source, and plug them into sensor fusion (8) in order to “nowcast” the current flu incidence (that would otherwise remain unknown for another week).

Such a sensor fusion system for flu nowcasting, using $d = 308$ sensors (flu proxies), is described in Chapter 4 of Farrow [2016]. In addition to the surveillance sensors described above (search volume for flu terms, site traffic metrics for flu pages, etc.), the measurement vector in this nowcasting system also uses a sensor that is trained to make predictions of wILI using a seasonal autoregression with 3 lags (SAR3). From the KF-SF equivalence established in Section 2, we can think of this SAR3 sensor as serving the role of something like a process model, in the underlying dynamical system.

While wILI itself is available at the US state level, the data source used to train each sensor may only be available at coarser geographic resolution. Thus, importantly, each sensor outputs a prediction at a different geographic resolution (which reflects the resolution of its corresponding data source). As an example, the number of visits to flu-related CDC pages are available for each US state separately; so for each US state, we train a separate sensor to predict wILI from CDC site traffic. However, counts for Wikipedia page visits are only available nationally; so we train just one sensor to predict national wILI from Wikipedia page visits.

Assuming unbiasedness of all the sensors, we construct the map $H$ in (2) so that its rows reflect the geography of the sensors. For example, if a sensor is trained on data that is available at the $i$th US state, then its associated row in $H$ is

$$(0, \ldots, 1, \ldots, 0);$$

and if a sensor is trained on data from the aggregate of the first 3 US states, then its associated row is

$$(w_1, w_2, w_3, 0, \ldots, 0),$$

for weights $w_1, w_2, w_3 > 0$ such that $w_1 + w_2 + w_3 = 1$, based on relative state populations; and so on. Figure 1 illustrates the setup in a simple example.
At a high-level, the constraints in (14) encode information about the measurement model (2). They also provide some kind of implicit regularization. Interestingly, as we will see later in Section 3.4, this can still be useful when used in addition to more typical (explicit) regularization.

How can we interpret these constraints? We give three interpretations, the first one specific to the flu forecasting setting, and the next two general.

**Flu interpretation.** In the flu nowcasting problem, recall, the map $H$ has rows that sum to 1, and they reflect the geographic level at which the corresponding sensors were trained (see Section 3.2). The constraints $H^T b_j = e_j$, $j = 1, \ldots, k$ can be seen in this case as a mechanism that accounts for the geographical hierarchy underlying the sensors. As a concrete example, consider the simplified flu nowcasting problem, with $k = 3$ states and $d = 8$ sensors. We have a 3-level hierarchy, where $x_1, x_2, x_3$ are part of the first region and $x_4, x_5$ are part of the second. The national level is at the root. Assuming all states have equal populations, the sensor map $H$ is

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1/2 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}.$$  

The national level is at the root. As for the sensors, we have one at each state, one at each region, and one at the national level. Assuming all states have equal populations, the sensor map $H$ is

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1/2 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}.$$  

The third line above can be interpreted as follows: an increase of 1 unit in sensor $z_3$, 1/3 units in $z_6$, and 1/5 units in $z_8$, holding all others fixed, should lead to an increase in 1 unit of our prediction for $x_3$. This is a natural consequence of the hierarchy in the sensor model (2), visualized in Figure 1. The first line can be read as: an increase of 1 unit in sensor $z_1$, 1/3 units in $z_6$, and 1/5 in $z_8$, with all others fixed, should not change our prediction for $x_3$. This is also natural, following from the hierarchy (i.e., such a change must have been propagated from $x_1$). The other lines are similar.

**Invariance interpretation.** The SF prediction (at time $t + 1$) is $\hat{x}_{t+1} = \hat{B}^T z_{t+1}$. To denoise (i.e., estimate the mean of) the measurement $z_{t+1}$, based on the model (2), we could use $\hat{z}_{t+1} = H \hat{x}_{t+1}$. Given the denoised $\hat{z}_{t+1}$, we could then refit our state prediction via $\hat{x}_{t+1} = \hat{B}^T \hat{H} \hat{x}_{t+1}$. But due to the constraint $H^T B = I$ (a compact way of expressing $H^T b_j = e_j$, for $j = 1, \ldots, k$), it holds that $\hat{x}_{t+1} = \hat{B}^T H \hat{x}_{t+1} = \hat{x}_{t+1}$. This is a kind of invariance property. In other words, we can go from estimating states, to refitting measurements, to refitting states, etc., and in this process, our state estimates will not change.

**Generative interpretation.** Assume $t \geq k$, and fix an arbitrary $j = 1, \ldots, k$ as well as $b_j \in \mathbb{R}^k$. The constraint $H^T b_j = e_j$ implies, by taking an inner product on both sides with $x_i$, $i = 1, \ldots, k$,

$$(H x_i)^T b_j = x_{ij}, \quad i = 1, \ldots, k.$$  

If we assume $x_i$, $i = 1, \ldots, k$ are linearly independent, then the above linear equalities are not only implied by $H^T b_j = e_j$, they are actually equivalent to it. Invoking the model (2), we may rewrite the
constraint $H^T b_j = e_j$ as

$$E(b_j^T z_i | x_i) = x_{ij}, \quad i = 1, \ldots, k.$$  \hspace{1cm} (15)

In the context of problem (14), this is a statement about a generative model for the data (as $z_i | x_i$ describes the distribution of the covariates conditional on the response). The reformulation in (15) shows that (14) constrains the regression estimator to have the correct conditional predictions, on average, on the data we have already seen $(x_i, z_i), i = 1, \ldots, k$. (Note here we did not have to use the first $k$ time points; any past $k$ time points would suffice.)

3.4 Modifications and equivalences

In the appendix, we show that two modifications of the basic SF formulation have direct equivalences in the regression perspective: namely, shrinking the empirical covariance in (13) towards the identity is equivalent to adding a ridge (squared $\ell_2$) penalty to the criterion in (14); and also, adding a null sensor at each state (one that always outputs 0) is equivalent to removing the constraints in (14). The latter equivalence here provides indirect but fairly compelling evidence that the constraints in the regression formulation (14) play an important role (under the model (2)): it says that removing them is equivalent to including meaningless null sensors, which intuitively should worsen its predictions.

4 Flu nowcasting application

In this section, we evaluate the performance of our methods for predicting one-week-ahead wILI over the flu seasons from 2013 to 2017 (a total of 140 weeks, or time points). Recall the nowcasting setup described in Section 3.2, with $k = 51$ states and $d = 308$ measurements. At time $t+1$, we derive an estimate $\hat{x}_{t+1}$ of the current wILI in the 51 US states, based on sensors $z_{t+1}$ (each is the output of an algorithm trained to predict wILI at a different geographic resolution), and past wILI data. Here, we consider four methods for computing $\hat{x}_{t+1}$: (a) SF, or equivalently constrained regression (14); (b) regression as in (14) but without constraints; (c) regression (14) without constraints and with a ridge penalty; and (d) SF with covariance shrinkage, or equivalently (14) with a ridge penalty. We choose all ridge tuning parameters at each time point using cross-validation (on only past data). To broaden the comparison, we also compare two other methods for predicting $\hat{x}_{t+1}$: (e) ARGO (autoregression with Google search data, Yang et al. [2015]), a prominent method from the flu nowcasting literature; and (f) random forests [Breiman, 2001], given access to the same set of features as our regression methods (a)–(d). As an error metric, we aggregate the wILI estimates up to the national level, and compute mean absolute error (MAE) to national wILI. Below we describe more details and results.

Missing data. Unfortunately, sensors are observed at not only varying geographic resolutions, but also varying temporal resolutions (since their underlying data sources are), and missing values occur. In our experiments, we choose to compute predictions using the regression perspective, and apply a simple mean imputation approach (using only past sensor data), before fitting the regression model.

Nowcasting results. Table 1 displays MAEs from the 6 methods considered in predicting national wILI, computed within each of the 4 seasons considered. The table also displays the mean absolute deviation (MAD) for each method in each season. We can see that SF with covariance shrinkage is among the top 2 most accurate methods for all 4 seasons, and is the most accurate for 2 of 4 seasons. We can also see that ridge regularization clearly helps: SF with shrinkage is more accurate than SF, and ridge regression is more accurate than regression. The appendix plots nowcasts along with the actual wILI curves for the 4 seasons.

SF-regression equivalence. After accounting for missingness (via mean imputation, as described above), we can compare the SF estimates from the regression formulation (14) and the original one (8), with the empirical covariance matrix (13) replacing the true one. The mean absolute difference between the two estimates (over all weeks and all seasons) is $1e-8$.

Role of constraints. As we can see in Table 1, the constraints in the regression formulation (14) (which come from its connection to SF) play an important role: SF is more accurate than regression, and SF with shrinkage is more accurate than ridge. Interestingly, random forests is not aware of the sensor hierarchy that encoded in these constraints, but still performs competitively. Of course, it does
<table>
<thead>
<tr>
<th>Season</th>
<th>Method</th>
<th>MAE</th>
<th>MAD</th>
<th>Season</th>
<th>Method</th>
<th>MAE</th>
<th>MAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2013-14</td>
<td>Sensor fusion</td>
<td>0.1887</td>
<td>0.0324</td>
<td>2013-14</td>
<td>Sensor fusion</td>
<td>0.2118</td>
<td>0.0347</td>
</tr>
<tr>
<td></td>
<td>Regression</td>
<td>0.2801</td>
<td>0.0473</td>
<td></td>
<td>Regression</td>
<td>0.8976</td>
<td>0.1504</td>
</tr>
<tr>
<td></td>
<td>Ridge</td>
<td>0.1549</td>
<td>0.0226</td>
<td></td>
<td>Ridge</td>
<td>0.1626</td>
<td>0.0293</td>
</tr>
<tr>
<td></td>
<td>SF + shrinkage</td>
<td>0.1267</td>
<td>0.0214</td>
<td></td>
<td>SF + shrinkage</td>
<td>0.1436</td>
<td>0.0246</td>
</tr>
<tr>
<td></td>
<td>ARGO</td>
<td>0.1302</td>
<td>0.0229</td>
<td></td>
<td>ARGO</td>
<td>0.0936</td>
<td>0.0112</td>
</tr>
<tr>
<td></td>
<td>Random forest</td>
<td>0.1059</td>
<td>0.0178</td>
<td></td>
<td>Random forest</td>
<td>0.1445</td>
<td>0.0216</td>
</tr>
<tr>
<td>2014-15</td>
<td>Sensor fusion</td>
<td>0.2009</td>
<td>0.0282</td>
<td>2016-17</td>
<td>Sensor fusion</td>
<td>0.2991</td>
<td>0.0510</td>
</tr>
<tr>
<td></td>
<td>Regression</td>
<td>0.3282</td>
<td>0.0511</td>
<td></td>
<td>Regression</td>
<td>0.5333</td>
<td>0.0869</td>
</tr>
<tr>
<td></td>
<td>Ridge</td>
<td>0.1831</td>
<td>0.0270</td>
<td></td>
<td>Ridge</td>
<td>0.2041</td>
<td>0.0317</td>
</tr>
<tr>
<td></td>
<td>SF + shrinkage</td>
<td>0.1504</td>
<td>0.0247</td>
<td></td>
<td>SF + shrinkage</td>
<td>0.1246</td>
<td>0.0221</td>
</tr>
<tr>
<td></td>
<td>ARGO</td>
<td>0.1876</td>
<td>0.0440</td>
<td></td>
<td>ARGO</td>
<td>0.1358</td>
<td>0.0319</td>
</tr>
<tr>
<td></td>
<td>Random forest</td>
<td>0.1590</td>
<td>0.0274</td>
<td></td>
<td>Random forest</td>
<td>0.1329</td>
<td>0.0234</td>
</tr>
</tbody>
</table>

Table 1: Mean absolute error (MAE) and mean absolute deviation (MAD) for nowcasting national wILI.

so because it combines the sensors flexibly (nonlinearly). This points to the fact that if we allowed SF to combined sensors more flexibly, then we should see even better results here; we return to this point in the discussion.

5 Extensions and discussion

In this paper, we studied connections between the Kalman filter, sensor fusion, and regression. We derived equivalences between the first two and latter two, discussed the general implications of our results, and gave a specific application to the problem of nowcasting the weekly influenza levels in the US, which motivated our work in the first place. We conclude with some potentially interesting extensions of the constrained regression formulation (14) of SF.

Sensor selection. The problem of selecting a small number of relevant sensors (on which to perform sensor fusion) among a possibly large number, which we can call sensor selection, is quite a difficult problem. Beyond this, measurement selection in the Kalman filter is a generally difficult problem. As far as we know, this is an active and relatively open area of research. On the other hand, in regression, variable selection is extremely well-studied, and \( \ell_1 \) regularization (among many other tools) is now very well-developed (see, e.g., Hastie et al. [2009, 2015]). Starting from the regression formulation for SF in (14), it would be natural to add to the criterion an \( \ell_1 \) or lasso penalty [Tibshirani, 1996] to select relevant sensors,

\[
\text{minimize } \sum_{i=1}^{t} (x_{ij} - b_j^T z_i)^2 + \lambda_j \|b_j\|_1 \\
\text{subject to } H^T b_j = e_j,
\]

where \( \|b_j\|_1 = \sum_{i=1}^{k} |b_{ji}|, j = 1, \ldots, k \). It is not clear (nor likely) that (16) has an equivalent SF formulation, but the exact equivalence when \( \lambda_j = 0 \) suggests that (16) could be a reasonable tool for sensor selection. Further, we perform a kind of process model selection with (16) by augmenting our measurement vector with multiple candidate process models, and penalizing only the corresponding coefficients. An empirical example is given in the appendix.

Nonlinearity. In the flu nowcasting problem, recall, the sensors were each outputs of a predictive model, which is trained to predict wILI from a particular flu proxy, using historical data. Practical benefits could arise, as we saw with random forests, by allowing SF to combine these sensors in a nonlinear manner. Starting from the regression formulation (14), we could extend this and solve

\[
\text{minimize } \sum_{i=1}^{t} (x_{ij} - f_j(z_i))^2 + \lambda_j P_j(f_j) \\
\text{subject to } \mathbb{E}[f_j(z_i) | x_i] = x_{ij},
\]

for \( j = 1, \ldots, k \). Here, each \( F_j \) is a function space and \( P_j \) is a penalty to be specified by the modeler (e.g., an RKHS and RKHS norm, respectively). Note that in (17), the constraints are inspired by the generative interpretation (15).
A Proofs and additional details

A.1 Proof of Theorem 1

We can write the sensor fusion update as

\[ \hat{P}_{t+1} = (\tilde{H}^T \tilde{R}^{-1}_{t+1} \tilde{H})^{-1} \]
\[ \hat{x}_{t+1} = \hat{P}_{t+1} \tilde{H}^T \tilde{R}^{-1}_{t+1} \tilde{z}_{t+1}, \]

where

\[ \hat{P}_{t+1} = (H^T R^{-1} H + \hat{P}^{-1}_{t+1})^{-1}. \]

By the Woodbury matrix identity, \((A + UCV^{-1}) = A^{-1} - A^{-1}U(C^{-1} + V A^{-1}U)^{-1}VA^{-1}, \) with \(A = \hat{P}^{-1}_{t+1}\) in our case, we get

\[ \hat{P}_{t+1} = \hat{P}_{t+1} - \hat{P}_{t+1} H^T (R + H \hat{P}_{t+1} H^T)^{-1} H \hat{P}_{t+1} \]
\[ = (I - \hat{P}_{t+1} H^T (R + H \hat{P}_{t+1} H^T)^{-1} H) \hat{P}_{t+1} \]
\[ = (I - K_{t+1} H) \hat{P}_{t+1}, \] \hspace{1cm} (18)

where recall, the Kalman gain \(K_{t+1}\) is defined in (6).

Now let us we rewrite the Kalman gain as

\[ K_{t+1} = \hat{P}_{t+1} H^T (R + H \hat{P}_{t+1} H^T)^{-1} \]
\[ = \hat{P}_{t+1} H^T R^{-1} (I + H \hat{P}_{t+1} H^T R^{-1})^{-1}, \]

so that

\[ K_{t+1} (I + H \hat{P}_{t+1} H^T R^{-1}) = \hat{P}_{t+1} H^T R^{-1}, \]
and after rearranging,

\[ K_{t+1} = (I - K_{t+1} H) \hat{P}_{t+1} H^T R^{-1}. \] \hspace{1cm} (19)

Putting (18) and (19) together, we get

\[ \hat{P}_{t+1} \tilde{H}^T \tilde{R}^{-1}_{t+1} \tilde{z}_{t+1} = (I - K_{t+1} H) \hat{P}_{t+1} (H^T R^{-1} z_{t+1} + \hat{P}^{-1}_{t+1} \tilde{x}_{t+1}) \]
\[ = (I - K_{t+1} H) \hat{P}_{t+1} H^T R^{-1} z_{t+1} + (I - K_{t+1} H) \tilde{x}_{t+1} \]
\[ = K_{t+1} z_{t+1} + (I - K_{t+1} H) \tilde{x}_{t+1} \]
\[ = \tilde{x}_{t+1} + K_{t+1} (z_{t+1} - H \tilde{x}_{t+1}), \]

which is exactly the Kalman filter prediction, completing the proof.

A.2 Derivation of (12)

We first make the EKF estimate precise. Let

\[ F_{t+1} = D f(\hat{x}_t), \]
\[ H_{t+1} = D h(\hat{x}_{t+1}), \]

and define

\[ \hat{x}_{t+1} = F_{t+1} \hat{x}_t, \] \hspace{1cm} (22)
\[ \hat{x}_{t+1} = \tilde{x}_{t+1} + K_{t+1} (z_{t+1} - h(\tilde{x}_{t+1})), \] \hspace{1cm} (23)

where \(K_{t+1} \in \mathbb{R}^{k \times d}\) is defined via

\[ \hat{P}_{t+1} = F_{t+1} \hat{P}_{t+1} F_{t+1}^T + Q, \] \hspace{1cm} (24)
\[ K_{t+1} = \hat{P}_{t+1} H_{t+1}^T H_{t+1}^{-1} (H_{t+1} \hat{P}_{t+1} H_{t+1}^T + R)^{-1}, \] \hspace{1cm} (25)
\[ \hat{P}_{t+1} = (I - K_{t+1} H_{t+1}) \hat{P}_{t+1}, \] \hspace{1cm} (26)

Note that (24)–(26) are exactly the same as (5)–(7), with \(F_{t+1}, H_{t+1}\) replacing \(F, H,\) respectively. Moreover, (22), (23) are nearly the same as (9), (11), with again \(F_{t+1}, H_{t+1}\) replacing \(F, H,\) except
that the residual in (23) is $z_{t+1} - h(\hat{x}_{t+1})$, and not $z_{t+1} - H_{t+1}\tilde{x}_{t+1}$, as would be analogous from (4).

Next, we make what we called the extended SF (ESF) estimate precise. Let $\tilde{z}_{t+1} \in \mathbb{R}^{d+k}$ be as in (11), let $\tilde{H}_{t+1} \in \mathbb{R}^{(d+k) \times k}$ be the rowwise concatenation of $H_{t+1}$ and $I \in \mathbb{R}^{k \times k}$, and $\tilde{R}_{t+1}$ be as in (2). Here, $F_{t+1}, H_{t+1}, P_{t+1}$ are as defined in (20), (21), (24), respectively. The ESF estimate is

$$\hat{x}_{t+1} = (\tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{H}_{t+1})^{-1} \tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{z}_{t+1}. \quad (27)$$

To see that (27) and (23) are equal, note that by following the proof of Theorem 1 directly, with $F_{t+1}, H_{t+1}$ in place of $F, H$, we get

$$(\tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{H}_{t+1})^{-1} \tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{z}_{t+1} = \hat{x}_{t+1} + K_{t+1}(z_{t+1} - H_{t+1}\tilde{x}_{t+1}).$$

Adding and subtracting $K_{t+1}h(\hat{x}_{t+1})$ to the right-hand side gives

$$(\tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{H}_{t+1})^{-1} \tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} (z_{t+1}, \tilde{x}_{t+1})$$

$$= \hat{x}_{t+1} + K_{t+1}(z_{t+1} - h(\hat{x}_{t+1})) + K_{t+1}(h(\hat{x}_{t+1}) - H_{t+1}\tilde{x}_{t+1})$$

$$= \hat{x}_{t+1} + K_{t+1}(z_{t+1} - h(\hat{x}_{t+1})) + (I - K_{t+1}H_{t+1})\tilde{P}_{t+1}H_{t+1}^T R^{-1}(h(\hat{x}_{t+1} - H_{t+1}\tilde{x}_{t+1})$$

$$= \hat{x}_{t+1} + K_{t+1}(z_{t+1} - h(\hat{x}_{t+1})) + \tilde{P}_{t+1}H_{t+1}^T R^{-1}(h(\hat{x}_{t+1} - H_{t+1}\tilde{x}_{t+1}),$$

where in the second line we used (19), and in the third we used (18). Rearranging gives

$$(\tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} \tilde{H}_{t+1})^{-1} \tilde{H}_{t+1}^T \tilde{R}_{t+1}^{-1} (z_{t+1} + H_{t+1}\tilde{x}_{t+1} - h(\hat{x}_{t+1}), \tilde{x}_{t+1}) = \hat{x}_{t+1} + K_{t+1}(z_{t+1} - h(\hat{x}_{t+1})),$$

which is precisely the desired conclusion, in (12).

### A.3 Empirical example of KF-SF equivalence

Figure 2 displays a simple empirical example demonstrating the KF-SF equivalence. We have only $k = 1$ state and $d = 1$ measurement, over 325 time points. We initialized $x_0 = 0$, and generated data from the process and measurement models (1), (3) with $F, H = I, Q = 0.25I$, and $R = 2I$ (where $I$ is the identity of dimension $1 \times 1$). We computed the KF estimate (3), (4), and the augmented SF estimate as in Theorem 1 over times $t = 1, \ldots, 325$. As the figure shows, they are visually identical. Numerically, the mean absolute difference between the two is 2e-16.

---

**Figure 2**: Simple dynamical linear system example comparing KF and SF.
A.4 Proof of Theorem\textsuperscript{2}

Let us denote $X \in \mathbb{R}^{t \times k}$ and $Z \in \mathbb{R}^{t \times d}$ the matrices of states and sensors, respectively, for the first $t$ time points. That is, $X$ has rows $x_i \in \mathbb{R}^k$, $i = 1, \ldots, t$ and $Z$ has rows $z_t \in \mathbb{R}^d$, $i = 1, \ldots, t$. Fix any $j = 1, \ldots, k$. Let $\hat{a}_j \in \mathbb{R}^d$ be the $j$th column of $\hat{R}_{t+1}^{-1}H(H^T \hat{R}_{t+1}^{-1}H)^{-1}$, and let $\hat{b}_j \in \mathbb{R}^d$ be the solution of (28), equivalently, the solution of

$$
\begin{align*}
\min_{b_j \in \mathbb{R}^d} & \quad \|X_j - Zb_j\|_2^2 \\
\text{subject to} & \quad H^T b_j = e_j,
\end{align*}
$$

(28)

where $X_j$ denotes the $j$th column of $X$. We will show that $\hat{a}_j = \hat{b}_j$.

The Lagrangian of problem (28) is

$$L(b_j, u_j) = \|X_j - Zb_j\|_2^2 + u_j^T (H^T b_j - e_j),$$

for a dual variable (Lagrange multiplier) $u_j \in \mathbb{R}^k$. Taking the gradient of the Lagrangian and setting it equal to zero at an optimal pair $(b_j, \hat{u}_j)$ gives

$$0 = Z^T (\hat{Z} \hat{b}_j - X_j) + H \hat{u}_j,$$

and rearranging gives

$$\hat{b}_j = (Z^T Z)^{-1} (Z^T X_j - H \hat{u}_j).$$

(29)

The dual solution $\hat{u}_j$ can be determined by plugging (29) into the equality constraint $H^T \hat{b}_j = e_j$, but for our purposes, the explicit dual solution is unimportant.

We will now show that $\hat{b}_j = \hat{R}_{t+1}^{-1}H \hat{\beta}_j$ for some $\hat{\beta}_j \in \mathbb{R}^k$. Write

$$\hat{R}_{t+1} = \frac{1}{t} (Z - XH^T)^T (Z - XH^T) + (1 - \alpha) I$$

$$= \frac{1}{t} (Z^T Z - X^T X + X^T H^T + H^T X H^T).$$

Then

$$\begin{align*}
\hat{R}_{t+1} \hat{b}_j &= \frac{1}{t} (Z^T Z \hat{b}_j - X^T X \hat{\beta}_j - X^T H^T \hat{\beta}_j + H^T X H^T \hat{\beta}_j) \\
&= \frac{1}{t} (Z^T X_j - H \hat{u}_j - X^T H \hat{\beta}_j - Z^T X_j + H^T X_j) \\
&= H \left( \frac{t}{\hat{\beta}_j} X^T X_j - \hat{a}_j - X^T Z \hat{b}_j \right),
\end{align*}$$

as desired, where in the second line we have used (29) and the constraint $H^T \hat{b}_j = e_j$.

Observe that $\hat{a}_j = \hat{R}_{t+1}^{-1}H \hat{\alpha}_j$ for some $\hat{\alpha}_j \in \mathbb{R}^k$, in particular, for $\hat{\alpha}_j$ defined to be the $j$th column of $(H^T \hat{R}_{t+1}^{-1}H)^{-1}$. Further,

$$e_j = H^T \hat{a}_j = H^T \hat{b}_j$$

the constraint on $\hat{a}_j$ holding by direct verification, and the constraint on $\hat{b}_j$ holding by construction in (28). That is,

$$H^T \hat{R}_{t+1}^{-1}H \hat{\alpha}_j = H^T \hat{R}_{t+1}^{-1}H \hat{\beta}_j,$$

and since $H^T \hat{R}_{t+1}^{-1}H$ is invertible, this leads to $\hat{\alpha}_j = \hat{\beta}_j$, and finally $\hat{a}_j = \hat{b}_j$, completing the proof.

A.5 Further SF-regression equivalences

A.5.1 More regularization: covariance shrinkage

Covariance shrinkage—which broadly refers to the technique of adding a well-conditioned matrix to a covariance estimate to provide stability and regularity—is widely used and well-studied in modern
multivariate statistics, data mining, and machine learning. As such, it would be natural to replace the empirical covariance matrix estimate \( \hat{\Sigma}_t \) for the measurement noise covariance by

\[
\hat{\Sigma}_{t+1} = \frac{1}{t} \sum_{i=1}^{t} (z_i - Hx_i)(z_i - Hx_i)^T + (1 - \alpha)I,
\]

for a parameter \( \alpha \in [0, 1] \). For sensor fusion in the flu nowcasting problem, this is considered (in some form) in [Farrow, 2016], and leads to significant improvements in nowcasting accuracy.

Our next result shows that when we use shrinkage as in (30) to estimate the measurement noise covariance in SF, this is equivalent to adding a ridge penalty in the regression formulation.

**Corollary 1.** Let \( \hat{\Sigma}_{t+1} \) be as in (30), for some value \( \alpha \in [0, 1] \). Consider the SF prediction at time \( t + 1 \), with \( R_t+1 \) in place of \( R_t \), denoted \( \hat{x}_{t+1} = \hat{B}^T z_{t+1} \). Then each column of \( \hat{B} \), denoted \( \hat{b}_j \in \mathbb{R}^d \), \( j = 1, \ldots, k \), solves

\[
\minimize_{b_j \in \mathbb{R}^d} \sum_{i=1}^{t} (x_{ij} - b_j^T z_i)^2 + \frac{t(1 - \alpha)}{\alpha} \|b_j\|^2 \tag{30}
\]

subject to \( H^T b_j = e_j \).

**Proof.** As before, let \( X \in \mathbb{R}^{t \times k} \) and \( Z \in \mathbb{R}^{t \times d} \) denote the matrix of states and sensors, respectively, over the first \( t \) time points. We can write \( R_{t+1} \) in (30)

\[
\frac{\alpha}{t} (Z - XH^T)^T (Z - XH^T) + (1 - \alpha)I = \frac{1}{t} (\hat{Z} - \hat{X}H^T)^T (\hat{Z} - \hat{X}H^T),
\]

where \( \hat{Z} \in \mathbb{R}^{(t+d) \times d} \) is the rowwise concatenation of \( \sqrt{\alpha/t}Z \) and \( \sqrt{1 - \alpha/I}I \), and \( \hat{X} \in \mathbb{R}^{(t+k) \times k} \) is the rowwise concatenation of \( \sqrt{\alpha/t}X \) and \( 0 \in \mathbb{R}^{k \times k} \) (the matrix of all 0s). Applying Theorem 2 to \( \hat{X}, \hat{Z} \), expanding the criterion in the regression problem, and then multiplying the criterion by \( t/\alpha \), gives the result.

### A.5.2 Less regularization: zero padding

In the opposite direction, we now show that we can modify SF and obtain an equivalent regression formulation with less regularization, specifically, without constraints.

**Corollary 2.** At each \( t = 1, 2, 3, \ldots \), suppose we augment our measurement vector by introducing \( k \) measurements that are identically zero, denoted \( \hat{z}_t = (z_t, 0) \in \mathbb{R}^{d+k} \). Suppose that we augment our measurement map accordingly, defining \( \hat{H} \in \mathbb{R}^{(d+k) \times k} \) to be the rowwise concatenation of \( H \) and the identity \( I \in \mathbb{R}^{k \times k} \). Consider running SF on this augmented system, using the empirical covariance to estimate \( R \), and let \( \hat{x}_{t+1} = \hat{B}^T z_{t+1} \) denote the SF prediction at time \( t + 1 \). Then each column of \( \hat{B} \), denoted \( \hat{b}_j \in \mathbb{R}^d \), \( j = 1, \ldots, k \), solves

\[
\minimize_{b_j \in \mathbb{R}^d} \sum_{i=1}^{t} (x_{ij} - b_j^T z_i)^2.
\]

**Proof.** Applying Theorem 2 to the augmented system gives the equivalent regression problem

\[
\minimize_{b_j \in \mathbb{R}^d, a_j \in \mathbb{R}^k} \sum_{i=1}^{t} (x_{ij} - b_j^T z_i - a_j^T 0)^2 \tag{30}
\]

subject to \( H^T b_j + a_j = e_j \).

The constraint is satisfied with \( a_j = e_j - H^T b_j \). But \( a_j \) has no effect on the criterion, so the constraint can be removed.

**Remark 4.** The analogous equivalence holds for covariance shrinkage and ridge regression. That is, in Corollary 2 if instead of the empirical covariance, we use \( \alpha \) times the empirical covariance plus \( (1 - \alpha)I \), then SF on the augmented system is equivalent to unconstrained ridge, at tuning parameter \( \frac{t(1 - \alpha)}{\alpha} \).
Figure 3: Nowcasts from the SF and regression methods plotted against underlying wILI values, for each of the 4 flu seasons considered.

A.6 More on flu nowcasting application

Figure 3 visually compares the nowcasts from the SF and regression methods, plotted against the true wILI curves, for the 4 flu seasons considered.

A.7 Example of process model selection

Here we give a simple empirical example of process model selection using the regression formulation of SF. We initialized $x_0 = 1$, and generated data according to

\[ x_t = 0.5x_{t-1} + 0.05 \sin(0.126t) + \delta_t, \]
\[ \delta_t = H x_t + \epsilon_t, \]

for $t = 1, \ldots, 200$. Here $H \in \mathbb{R}^{4 \times 1}$ is simply the column vector of all 1s, and the noise is drawn as $\delta_t \sim N(0, 0.01)$, $\epsilon_t \sim N(0, I)$, independently, over $t = 1, \ldots, 150$.

The prediction setup is as follows. At each time $t + 1$, when making a prediction of $x_{t+1}$, we observe all past states $x_i$, $i = 1, \ldots, t$ and all measurements $z_i$, $i = 1, \ldots, t + 1$. We fit 5 different candidate process models to past state data:

1. linear autoregression;
2. quadratic autoregression;
3. spline regression on time;
4. sine regression on time;
5. cosine regression on time.
To be clear, models 1 and 2 regress $x_i$ on $x_{i-1}$ and $x_{i-1}^2$, respectively, over $i = 1, \ldots, t$. Models 3–5 regress $x_i$ on a spline, sine, and cosine transformation of $i$, respectively, over $i = 1, \ldots, t$. The sine and cosine transformations are given the true frequency. The spline is a cubic smoothing spline (with a knot at every data point) and its tuning parameter is chosen by cross-validation (using only the past data). After being fit, we use each of the candidate process models 1–5 to make a prediction of $x_{t+1}$, given $z_{t+1}$. We take this as its output.

For $t = 151, \ldots, 200$, we define $\tilde{z}_t \in \mathbb{R}^9$ to be the measurement vector $z_t \in \mathbb{R}^4$ augmented with the outputs of the 5 candidate process models as described above (the burn-in period of 150 time points ensures that the candidate process models have enough training data to make reasonable predictions). Figure 4 shows the outputs from these models over the last 50 time points.

Finally, in the last 50 time points, to get an assimilated prediction of $\hat{x}_{t+1}$ at each time $t+1$, we solve the constrained regression problem with a lasso penalty (16), using cross-validation to select $\lambda$ (again, using only past data). Further, we penalize only the coefficients of the candidate process models (not the pure measurements). Table 2 shows the median of the coefficients over the last 50 time points (in this table, the coefficients for the pure measurement sensors are aggregated as one). We see that the lasso tends to select the linear and sine sensors, as expected (because these two make up the true dynamical model), and places a small weight on the spline sensor (which is flexible, and can mimic the contribution of the sine sensor).

<table>
<thead>
<tr>
<th>Median Coefficient</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Spline</th>
<th>Sine</th>
<th>Cosine</th>
<th>Measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.643</td>
<td>0.000</td>
<td>0.094</td>
<td>0.189</td>
<td>0.000</td>
<td>0.0175</td>
</tr>
</tbody>
</table>

Table 2: Simple process model selection example: median regression coefficients for the sensors, over the last 50 time points.

References


