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Kalman Filter, Sensor Fusion, and Constrained Regression: Equivalences and Insights

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Kalman Filter and Sensor Fusion

Kalman Filter Sequential estimation via two-step process

$$x_t = F x_{t-1} + \delta_t, \quad \delta_t \sim \mathcal{N}(0, Q)$$
$$z_t = H x_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, R)$$

Process model predicts intermediate state $\bar{x}_{t+1} = F\hat{x}_t$, then perform update with observed measurement

$$\hat{x}_{t+1} = \bar{x}_{t+1} + K_{t+1}(z_{t+1} - H\bar{x}_{t+1}),$$

where K_{t+1} is the Kalman gain.

Sensor Fusion Send process noise $Q \to \infty$ (flat prior), state estimate is regression of measurement z on H:

$$\hat{x}_{t+1} = (H^T R^{-1} H)^{-1} H^T R^{-1} z_{t+1}$$

- Possible to recover state dynamics exactly

Contributions

- **KF is equivalent to augmented SF**

  Augment $\tilde{z}_{t+1} = (z_{t+1}, \bar{x}_{t+1})$

  $$\tilde{H} = \begin{bmatrix} H \\ I_k \end{bmatrix} \qquad \tilde{R}_{t+1} = \begin{bmatrix} R & 0 \\ 0 & \bar{P}_{t+1} \end{bmatrix}$$

  $$(\tilde{H}^T \tilde{R}_{t+1}^{-1} \tilde{H})^{-1} \tilde{H}^T \tilde{R}_{t+1}^{-1} \tilde{z}_{t+1} = \bar{x}_{t+1} + K_{t+1}(z_{t+1} - H\bar{x}_{t+1})$$

- **SF is equivalent to regression with constraints**

  Regress states x on measurements z, subject to interpretable equality constraint $H^T B = I$

- **Extensions given by the regression viewpoint**
  - ℓ_2 penalty $\iff$ covariance shrinkage
  - ℓ_1 penalty $\to$ sensor (or process model) selection
  - gradient boosting to jointly fit sensors

- **SF for state-of-the-art influenza prediction**

Regression Equivalence

SF prediction is $\hat{x}_{t+1} = \hat{B}^T z_{t+1}$, where

$$\hat{B}^T = (H^T \hat{R}_{t+1}^{-1} H)^{-1} H^T \hat{R}_{t+1}^{-1},$$

and $\hat{R}_{t+1}$ is the empirical covariance (from observed states).

Each column of $\hat{B}$, denoted $\hat{b}_j \in \mathbb{R}^d$, $j = 1, \ldots, k$, solves

$$\underset{b_j \in \mathbb{R}^d}{\text{minimize}} \sum_{i=1}^{t} (x_{ij} - b_j^T z_i)^2 \quad \text{subject to} \quad H^T b_j = e_j,$$

where $e_j \in \mathbb{R}^d$ is the jth standard basis vector.

A $(1-\alpha)/\alpha \|b_j\|_2^2$ penalty $\iff$ covariance shrinkage:

$$\hat{R}_{t+1} = \frac{\alpha}{t} \sum_{i=1}^{t} (z_i - Hx_i)(z_i - Hx_i)^T + (1-\alpha) I.$$

Extensions

Sensor selection Learn relevant sensors (or process model). With same constraint $H^T b_j = e_j$,

$$\underset{b_j \in \mathbb{R}^d}{\text{minimize}} \frac{1}{t} \sum_{i=1}^{t} (x_{ij} - b_j^T z_i)^2 + \lambda_j \|b_j\|_1$$

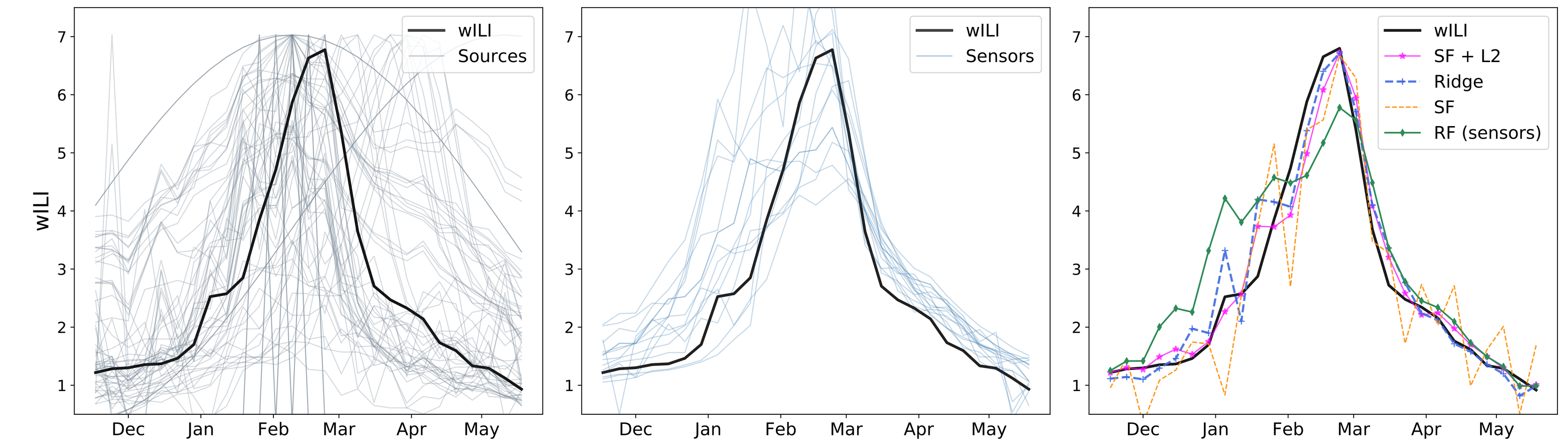
Gradient Boosting Can iterate between sensor fitting and sensor fusion. At time i, denote u_i as the data source and $y_i = Hx_i$. Set $x_i^{(0)}$, then repeat for $b = 1, \ldots, B$

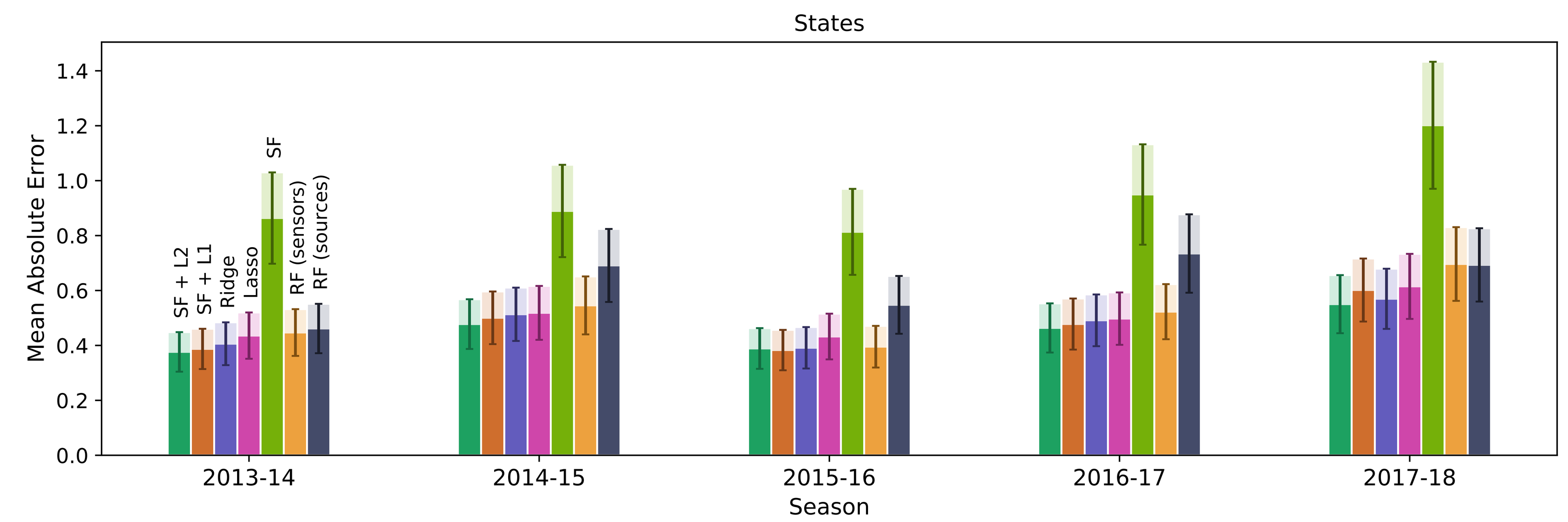
- For each source j, set $y_{ij}^{(b-1)} = (Hx^{(b-1)})_{ij}$
- Fit response $\{y_{ij} - y_{ij}^{(b-1)}\}_{i=1}^t$ with $\{u_{ij}\}_{i=1}^t$ to get $z_{ij}^{(b)}$
- For each state j, run SF on $\{x_{ij} - x_{ij}^{(b-1)}\}_{i=1}^t$ with $\{z^{(b)}\}_{i=1}^t$ to get intermediate state fit $\bar{x}_{ij}^{(b)}$
- Update total state fit $x_{ij}^{(b)} = x_{ij}^{(b-1)} + \eta \bar{x}_{ij}^{(b)}$

Application: Influenza Nowcasting

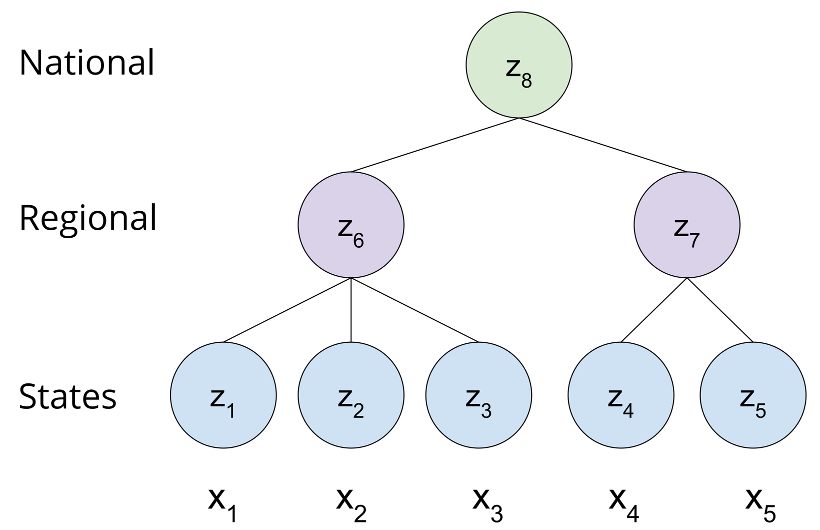
Nowcast weekly flu incidence in 51 US states

- Track *weighted Influenza-like Illness*, available after 1 week
- 308 sensors fitted with digital surveillance sources observed at different geographic resolutions, e.g.
  - web searches with flu-related terms
  - tweets indicating flu infection
  - visits to Wikipedia or CDC flu pages
- Use auto-regressive sensor as process model analogue





Role of Constraints



$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

For state x_3:

$$b_{33} + 1/3\, b_{36} + 1/5\, b_{38} = 1$$
$$b_{34} + 1/3\, b_{37} + 1/5\, b_{38} = 0$$
$$\ldots$$