Problem 1

Applying the Expectation Additivity and Scaling Rules yields the following:

\[ E(U) = E(X + Y + Z) = E(X) + E(Y) + E(Z) = 2 + 3 + 6 = 11 \]

\[ E(V) = E(X + Y + YZ) = E(X) + E(Y) + E(YZ) = 2 + 3 - 3 = 2 \]

\[ E(W) = E(XY - 3XZ + 2Y^2 - 6YZ - X) = E(XY) - 3E(XZ) + 2E(Y^2) - 6E(YZ) - E(X) = 2 - 12 + 20 - 6(-3) - 2 = 26 \]

Problem 2

For any value of \( X \), \( \mathbb{I}\{X \leq k\} \) equals 1 for all values of \( k \) from 1 to \( X \) and 0 otherwise. Therefore, exactly \( X \) ones and \( M - X \) zeros enter the sum. Hence, both sides of the equation are equal.

Applying the linearity of expectation yields:

\[ E(X) = \sum_{i=1}^{M} P(X \geq i) \]
Problem 3

If the three line segments are to form a triangle, the longest side cannot be greater than 1/2. Hence, we require that the following event occurs

\[ A = \{X < 1/2, Y > 1/2, Y - X < 1/2\} \bigcup \{Y < 1/2, X > 1/2, X - Y < 1/2\}. \]

These sub-events – call them \( A_1 \) and \( A_2 \) – are disjoint, so it is enough to compute the probabilities of each separately. Because they are symmetrically defined and the joint distribution of \((X, Y)\) is symmetric, the two events must also have the same probability.

There are several approaches one might take. The following one uses the area of a graph to show the probability. The joint PDF \( f_{XY} \) equals the constant 1 over the unit square and 0 elsewhere:

\[ f_{XY}(u,v) = 1_{[0,1]}(u)1_{[0,1]}(v). \]

\(P(A)\) thus equals the area of the set of \((u, v)\) satisfying the conditions of the events. This corresponds to the two shaded triangles in the figure below, each of which has area 1/8. Hence, \( P(A) = 1/4 \).

You can also compute the probability by integrating over joint density of \( X \) and \( Y \).

\[ P(\{X < 1/2, Y > 1/2, Y - X < 1/2\}) = \int_0^{1/2} \int_{1/2}^1 1_{y-x<1/2} dxdy = 1/8. \]
Problem 4

We condition the expected time on the door he pick for the first time. If he picks the first door, the miner will get out and the conditional expected $E(\text{Time}|D_1) = 3$.

Otherwise, when he choose $D_i (i = 2, 3, 4)$, he will spend $T_i$ to return to the mine and restart the process of escape. In this case, aside from the time spent on returning $T_i$, he still needs the same amount of time to escape as from the beginning. So the conditional expected time here would be $E(\text{Time}|D_i) = T_i + E(\text{Time})$.

In all, by the law of total expectation,

$$E(\text{Time}) = E(\text{Time}|D_1)P(D_1) + E(\text{Time}|D_2)P(D_2) + E(\text{Time}|D_3)P(D_3) + E(\text{Time}|D_4)P(D_4)$$

$$= 3 \times (1/4) + (4 + E(\text{Time})) \times (1/2) + (2 + E(\text{Time})) \times (1/8) + (1 + E(\text{Time})) \times (1/8)$$

Solving it for $E(\text{Time})$, and the expected time to escape is 12.5 hours.

Problem 5

Notation: $H_i$: player $i$ hits the target; $M_i$: player $i$ misses the target.

Number of shots: $N_s$

First, let’s consider $\mu_1$. Conditioning on the first step (1st player hits or misses the target),

$$\mu_1 = E(N_s|H_1)P(H_1) + E(N_s|M_1)P(M_1).$$

Two conditional expectations on the RHS is not trivial.

$$E(N_s|M_1) = 1 + \mu_2.$$ 

It’s because in this case the first player has missed, the 2nd player will shot. Now we need the same number of shots as when the game start from 2nd player, in addition to the 1 shot that have happened.

Now to compute $E(N_s|H_1)$, we condition on the second step (2nd player hits or misses the target),

$$E(N_s|H_1) = E(N_s|H_1, H_2)P(H_2) + E(N_s|H_1, M_2)P(M_2) = 2q + (2 + \mu_1)(1 - q).$$

In the last equation, for $E(N_s|H_1, M_2)$, the second player misses the target, so the first player restart the game, and we still need the same amount of shots as from the beginning by first player $\mu_1$, in addition to the 2 shots that have happened.

Plug in the conditional expectations, and we get:

$$\mu_1 = (2q + (2 + \mu_1)(1 - q))p + (1 + \mu_2)(1 - p)$$
We can apply similar procedure to $\mu_2$, and we get:

$$\mu_2 = (2p + (2 + \mu_2)(1 - p))q + (1 + \mu_1)(1 - q)$$

Solving the above two equation for $\mu_1$ and $\mu_2$, we can get the explicit expression in terms of $p$ and $q$. [Not required]

**Number of hits: $N_h$**

The idea is very similar with computing number of shots. Some number in conditional expectation is different. For $h_1$,

$$h_1 = E(N_h|H_1)P(H_1) + E(N_h|M_1)P(M_1)$$

$$= (E(N_s|H_1, H_2)P(H_2) + E(N_s|H_1, M_2)P(M_2))p + h_2(1 - p)$$

$$= (2q + (1 + h_1)(1 - q))p + h_2(1 - p)$$

Here $E(N_h|M_1) = h_2$ and $E(N_s|H_1, M_2) = 1 + h_1$ because $N_h$ is the number of HITS. Similarly for $h_2$,

$$h_2 = (2p + (1 + h_2)(1 - p))q + h_1(1 - q)$$

Again, solving the above two equation for $h_1$ and $h_2$, we can get the explicit expression in terms of $p$ and $q$. [Not required]