1. Suppose you have a Markov chain with \( k \) states and initial probabilities \( \pi \), i.e. 
\[ P(X_0 = 0) = \pi_0, \text{ and so on.} \] 
The Markov chain has transition matrix \( P \). Express the probability of a path,
\[ P(X_0 = i_0, X_1 = i_1, \ldots, X_m = i_m) \]
in terms of these quantities.

- Balls are thrown one at a time and must land in one of \( b \) boxes, with equal probability. Let \( X = (X_0, X_1, X_2, \ldots) \) be a stochastic process where \( X_n \) denotes the number of occupied boxes after the \( n \)th throw. Find the state space and transition probability matrix for the chain \( X \).

2. Consider the following model for the diffusion of gas. Suppose that \( M \) molecules are distributed between two chambers that are separated by a permeable boundary. At each time \( n \), one of the molecules – with all equally likely to be chosen – crosses the boundary from one chamber to the other.

(i) Give a brief (non-mathematical) argument that this a Markov chain.

(ii) Is this process time homogeneous or time inhomogeneous? Briefly justify your answer.

(iii) Describe the state space of this chain and give the transition probabilities.

3. A spider is hunting a fly, and the fly is trying to survive. The spider starts in location 1 and moves between locations 1 and 2 according to the Markov transitions
\[
P^S = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}. \tag{1}
\]
The fly starts in location 2 and moves between the locations with transitions
\[
P^F = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}. \tag{2}
\]
The spider and fly move independently. The hunt ends if the two ever land on the same location, in which case the fly is eaten.

(i) Show that this progress of the hunt can be described (except for knowing at which location the hunt ends) by a three-state Markov chain.

(ii) Find the transition probabilities for this chain.

4. Write out the transition matrix, and draw the transition diagram, for a finite Markov chain that satisfies the following properties:
• The chain has one absorbing state.
• The chain has exactly three recurrent classes.
• The chain has at least one transient state.
• All recurrent states in the chain are accessible from all transient states.

5. A Markov chain has transition matrix:

\[ P = \begin{bmatrix} p & 1 - p \\ 1 - p & p \end{bmatrix}. \]

Use mathematical induction to prove that

\[ P^n = \begin{bmatrix} 0.5 + 0.5(2p - 1)^n & 0.5 - 0.5(2p - 1)^n \\ 0.5 - 0.5(2p - 1)^n & 0.5 + 0.5(2p - 1)^n \end{bmatrix}. \]