1. Suppose you have a Markov chain with \( k \) states and initial probabilities \( \pi \), i.e. \( \mathbb{P}(X_0 = 0) = \pi_0 \), and so on. The Markov chain has transition matrix \( P \). Express the probability of a path,
\[
\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_m = i_m)
\]
in terms of these quantities.

**Solution:** Let us denote the \((i,j)\)th component of transition matrix \( P \) by \( P_{i,j} \).

By the chain rule,
\[
\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_m = i_m) = \mathbb{P}(X_m = i_m | X_{m-1} = i_{m-1}, \ldots, X_0 = i_0) \mathbb{P}(X_0 = i_0, \ldots, X_{m-1} = i_{m-1})
\]
where the second equality follows by the Markov property. In a similar fashion, the second term can be further decomposed into
\[
\mathbb{P}(X_0 = i_0, \ldots, X_{m-1} = i_{m-1}) = \prod_{k=1}^{m-1} \mathbb{P}(X_k = i_k | X_{k-1} = i_{k-1}) \mathbb{P}(X_0 = i_0)
\]
where the second equality follows by the time-homogeneity and \( \mathbb{P}(X_0 = i_0) = \pi_{i_0} \).

Therefore,
\[
\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_m = i_m) = \prod_{k=1}^{m-1} P_{i_{k-1},i_k} \times \pi_{i_0},
\]

• Balls are thrown one at a time and must land in one of \( b \) boxes, with equal probability. Let \( X = (X_0, X_1, X_2, \ldots) \) be a stochastic process where \( X_n \) denotes the number of occupied boxes after the \( n \)th throw. Find the state space and transition probability matrix for the chain \( X \).

**Solution:** There can be anywhere from 0 to \( b \) boxes occupied. So, \( S = \{0, \ldots, b\} \).

In state \( i \), the thrown ball lands in an occupied box with probability \( i/b \) and in an unoccupied box with probability \( 1 - i/b \).

Hence,
\[
P_{i,j} = \begin{cases} 
  i/b & \text{if } j = i \\
  1 - i/b & \text{if } j = i + 1 \\
  0 & \text{otherwise.}
\end{cases}
\]
2. Consider the following model for the diffusion of gas. Suppose that $M$ molecules are distributed between two chambers that are separated by a permeable boundary. At each time $n$, one of the molecules – with all equally likely to be chosen – crosses the boundary from one chamber to the other.

(i) Give a brief (non-mathematical) argument that this a Markov chain.

**Solution:** The system is described by the number of molecules in the two chambers. Because there are always $M$ molecules in total, we can describe the state of the system by the number of molecules in the left chamber. (Either chamber will do.)

The change in the system at any time depends only on how many molecules are in the left (or alternatively the right) chamber. Whatever the history, we select one ball to switch chambers, and the distribution of that change is a function only of how many balls there are in each chamber. Thus, heuristically, the Markov property holds.

(ii) Is this process time homogeneous or time inhomogeneous? Briefly justify your answer.

**Solution:** The system is time homogeneous because the rules for determining the transitions do not change over time and because the total number of balls remains fixed.

(iii) Describe the state space of this chain and give the transition probabilities.

**Solution:** At any time, there are only two possibilities: either we pick a ball in the left chamber which moves to the right chamber, or we pick a ball in the right chamber that moves to the left chamber. Thus, if we are in state $k$ (meaning $k$ balls in the left chamber), we have probability $k/M$ of picking a ball in the left chamber and thus going to state $k - 1$, and we have probability $(M-k)/M = 1 - k/M$ of picking a ball in the right chamber and moving to state $k + 1$. Hence, the state space is $S = \{0, 1, 2, \ldots, M\}$ and the transition probability is

$$P_{k \ell} = \begin{cases} 
\frac{k}{M} & \text{if } \ell = k - 1 \\
\frac{M-k}{M} & \text{if } \ell = k + 1 \\
0 & \text{otherwise.}
\end{cases}$$

This completely specifies the transition probability matrix.

3. A spider is hunting a fly, and the fly is trying to survive. The spider starts in location 1 and moves between locations 1 and 2 according to the Markov transitions

$$P^S = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

(1)
The fly starts in location 2 and moves between the locations with transitions

\[ P^F = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}. \]  

(2)

The spider and fly move independently. The hunt ends if the two ever land on the same location, in which case the fly is eaten.

(i) Show that this progress of the hunt can be described (except for knowing at which location the hunt ends) by a three-state Markov chain.

**Solution:** The state of the system is specified by the positions of the spider and the fly. If they meet at any time, however, it does not matter to us where they are, so we can summarize the system’s configuration with three possible states:

i. Spider at location 1, Fly at location 2
ii. Spider at location 2, Fly at location 1
iii. Spider and Fly at location 1 or Spider and Fly at location 2.

Once the system reaches state 3, it stays there, so 3 is an absorbing state. We can compute the other elements of the transition matrix by using the individual transition matrices of the spider and fly. Because the spider and fly move according to a Markov chain and because state 3 is absorbing, this process satisfies the Markov property as well. Because the individual movement processes are time homogeneous so is this chain.

(ii) Find the transition probabilities for this chain.

**Solution:** The transition matrix is given by

\[
P = \begin{bmatrix} 0.42 & 0.12 & 0.46 \\ 0.18 & 0.28 & 0.54 \\ 0 & 0 & 1 \end{bmatrix}
\]

which is what we needed to find.

To see \( P \) contains these values, let \( H \) describe this chain, \( S \) describe the spider’s position (also a markov chain), and \( F \) describe the fly’s position. Let \( P^S \) and \( P^F \) be the spider and fly transition matrices as given. Let \( i, j \in \{1, 2\} \). Then,

\[
P_{ij} = \mathbb{P}(H_{n+1} = j|H_n = i)
= \mathbb{P}(S_{n+1} = j \text{ and } F_{n+1} = 3 - j|S_n = i \text{ and } F_n = 3 - i)
= \mathbb{P}(S_{n+1} = j|S_n = i) \mathbb{P}(F_{n+1} = 3 - j|F_n = 3 - i)
= P^S_{ij} P^F_{3-i,3-j}.
\]

Similarly, \( P_{3i} = 0 \) and

\[
P_{33} = \mathbb{P}(S_{n+1} = i \text{ and } F_{n+1} = i|S_n = i \text{ and } F_n = 3 - i) 
+ \mathbb{P}(S_{n+1} = 3 - i \text{ and } F_{n+1} = 3 - i|S_n = i \text{ and } F_n = 3 - i)
= P^S_{ii} P^F_{3-i,i} + P^S_{i,3-i} P^F_{3-i,3-i}.
\]
4. Write out the transition matrix, and draw the transition diagram, for a finite Markov chain that satisfies the following properties:

- The chain has one absorbing state.
- The chain has exactly three recurrent classes.
- The chain has at least one transient state.
- All recurrent states in the chain are accessible from all transient states.

**Solution:** Consider the transition matrix

\[ P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}
\end{bmatrix}. \]

Denoting the states as \( S = \{1, 2, 3, 4, 5, 6\} \) we see that \( \{1\}, \{2, 3\}, \text{ and } \{4, 5\} \) are recurrent classes, \( \{6\} \) is a transient state, and the remaining two requirements are satisfied.

![Transition Diagram](image)

**Figure 1:** Transition diagram

5. A Markov chain has transition matrix:

\[ P = \begin{bmatrix}
p & 1 - p \\
1 - p & p
\end{bmatrix}. \]

Use mathematical induction to prove that

\[ P^{(n)} = \begin{bmatrix}
0.5 + 0.5(2p - 1)^n & 0.5 - 0.5(2p - 1)^n \\
0.5 - 0.5(2p - 1)^n & 0.5 + 0.5(2p - 1)^n
\end{bmatrix}. \]

**Solution:** We first check the base case: For \( n = 1 \), we can verify that \( P^{(1)} = P \) as desired.
For the inductive case we first assume that for $n = k$ we have

$$P^{(k)} = \begin{bmatrix} 0.5 + 0.5(2p - 1)^k & 0.5 - 0.5(2p - 1)^k \\ 0.5 - 0.5(2p - 1)^k & 0.5 + 0.5(2p - 1)^k \end{bmatrix}.$$ 

We now need to use this and verify the claim is true for $n = k + 1$. By Chapman-Kolmogorov we know that

$$P^{(k+1)} = P^{(k)} \ast P.$$ 

Now we verify the claim entry-wise.

$$P^{(k+1)}_{11} = \sum_{j=1}^{2} P^{(k)}_{1j} \ast P_{j1}$$

$$= (0.5 + 0.5(2p - 1)^k)p + (0.5 - 0.5(2p - 1)^k) \ast (1 - p)$$

$$= 0.5 + 0.5(2p - 1)^{k+1},$$

as desired. And

$$P^{(k+1)}_{12} = \sum_{j=1}^{2} P^{(k)}_{1j} \ast P_{j2}$$

$$= (0.5 + 0.5(2p - 1)^k)(1 - p) + (0.5 - 0.5(2p - 1)^k) \ast p$$

$$= 0.5 - 0.5(2p - 1)^{k+1},$$

as desired. The remaining two entries are identical establishing the inductive claim. By mathematical induction we conclude that the statement is true for all $n \geq 1$. 