Questions are worth four points each, unless otherwise noted.

1. Imagine a random walk on the integers that moves up to the next larger integer with probability 0.6, and moves down with probability 0.4. If the walk starts at zero, what is the probability it reaches 2 before reaching −2? You can use MATLAB if necessary.

2. This question is similar to one you had on the previous HW. A Markov chain \( \{X_n, n \geq 0\} \) with states 0, 1, 2 has the transition probability matrix

\[
\begin{pmatrix}
0.5 & 0.3 & 0.2 \\
0 & 1/3 & 2/3 \\
1/2 & 0 & 1/2
\end{pmatrix}
\]

If \( P(X_0 = 0) = P(X_0 = 1) = 1/4 \), find \( E(X_3) \).

3. Balls are thrown one at a time and must land in one of \( b \) boxes, with equal probability. Let \( X = (X_0, X_1, X_2, \ldots) \) be a stochastic process where \( X_n \) denotes the number of occupied boxes after the \( n^{th} \) throw. Find the state space and transition probability matrix for the chain \( X \).

4. A Markov chain has transition matrix:

\[
P = \begin{bmatrix}
p & 1-p \\1-p & p
\end{bmatrix}.
\]

Use mathematical induction to prove that

\[
P^{(n)} = \begin{bmatrix}
0.5 + 0.5(2p - 1)^n & 0.5 - 0.5(2p - 1)^n \\
0.5 - 0.5(2p - 1)^n & 0.5 + 0.5(2p - 1)^n
\end{bmatrix}.
\]

5. Let \( X = (X_n)_{n \geq 0} \) be a time homogeneous Markov chain on the state space \( S = \{1, 2, 3, 4, 5, 6, 7\} \). Assume that the initial state \( X_0 \) is either 2 or 3 with respective probability 1/2. Assume also that the transition probability matrix is given by

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.1 & 0 & 0.4 & 0 & 0.4 & 0 & 0.1 \\
0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0.5 & 0 & 0.2 \\
0 & 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0 & 0.25 & 0.75 & 0 \\
0 & 0 & 0 & 0 & 0.6 & 0.4 & 0
\end{bmatrix}
\]
The rows and columns are labeled with states in numeric order 1..7. We also have a partially complete $P^{(7)}$:

$$
P^{(7)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.329 & 0 & 0.021 & 0 & 0.056 & 0.24 & 0.354 \\
0.364 & 0.034 & 0 & 0.046 & 0 & 0.278 & 0.278 \\
0.404 & 0 & 0.011 & 0 & 0.028 & 0.235 & 0.322 \\
0.208 & 0.023 & 0 & 0.032 & 0 & 0.349 & 0.388 \\
0 & 0 & 0 & 0 & 0 & 0.444 & 0.556 \\
0 & * & 0 & 0 & 0 & * & 0.555
\end{bmatrix}
$$

(a) [2 points] Fill in the two missing elements of $P^{(7)}$.

(b) [2 points] Find $P(X_{n+2} = 5|X_n = 3)$.

(c) [3 points] Find $P(X_8 = 1)$. Note: There are less and more tedious ways to do this. Show your setup for the calculation and consider what information you really need.

(d) [1 point] Find $P(X_{117} = 4|X_0 = 2)$.

(e) [1 point] Can state 2 be reached from state 4? How do you know?

(f) [2 points] Describe in words the behavior of the chain, and its behavior in the long-term.

6. Write out the transition matrix, and draw the transition diagram, for a finite Markov chain that satisfies the following properties:

- The chain has no absorbing states.
- The chain has exactly two recurrent classes.
- The chain has at least one transient state.
- All recurrent states in the chain are accessible from all transient states.