Questions are worth four points each, unless otherwise noted.

1. A transition probability matrix always has row-sums equal to 1. Such matrices are called (row)-stochastic. A transition probability matrix that has row and column sums equal to 1 is called \(\text{doubly}\) stochastic, that is

\[
\sum_i P_{ij} = 1
\]

for all \(j\). If such a chain is both irreducible and aperiodic and consists of \(M\) states \(\{0, 1, \ldots, M-1\}\), then show that there is a limiting distribution and that the limiting distribution has entries:

\[
\pi_j = \frac{1}{M}, \quad j = \{0, 1, \ldots, M\}.
\]

2. A DNA nucleotide has any one of four values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability \(1 - 3\alpha\) and if it does change, it takes on one of the other three values with equal probability \(0 < \alpha < 1/3\).

   (a) Show that \(P_{11}^n = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n\).
   (b) What is the long-run proportion of time the chain is in each state?

3. Write out the transition diagram, transition matrix, and limiting distribution for one Markov chain that satisfies all of the following requirements:
   
   (a) There is exactly one transient class.
   (b) All transient states have period three.
   (c) There is exactly one recurrent class.
   (d) There are at least two recurrent states.
   (e) The chain has a unique limiting distribution.

4. Define a Markov chain on \(\{0, \ldots, 5\}\) with transition probability matrix

\[
P = \begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 2/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6
\end{bmatrix}
\]

Tell me everything you can about this chain. Here are some particular questions you should answer but please try to think through and write down other questions that you
can answer: What are the communicating classes? Recurrent or transient? What is the 
limiting behavior of the chain corresponding to each class? What does this say about 
the long-run behavior of the chain itself? What can we say about the transient class? 
What is $P^{(\infty)}$?

5. Define a Markov chain on $\{0, \ldots, 5\}$ with transition probability matrix

$$
P = \begin{bmatrix}
0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0.25 & 0.5 & 0.25 & 0 & 0 & 0 \\
0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.25 & 0.75 & 0 \\
0 & 0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0.25 & 0.25 & 0 & 0.5 \\
\end{bmatrix}
$$

Tell me everything you can about this chain.

6. Define a Markov chain on $\{0, \ldots, 5\}$ with transition probability matrix

$$
P = \begin{bmatrix}
0 & 0 & 0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.25 & 0.75 \\
0 & 0 & 0 & 0 & 0.25 & 0.75 \\
0.8 & 0.2 & 0 & 0 & 0 & 0 \\
0.8 & 0.2 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Tell me everything you can about this chain.

7. Let $Y_n$ be the sum of $n$ independent rolls of a fair die. Find

$$
\lim_{n \to \infty} P(Y_n \text{ is a multiple of 13}).
$$

**Hint:** Define an appropriate Markov chain and then use the expression from Question 1.

8. Suppose a Markov chain has a finite state space $S$.

(a) Let $j, k \in S$ and assume $k$ is accessible from $j$, but $j$ is not accessible from $k$. 
TRUE or FALSE: State $j$ must be transient. Explain your answer briefly.

(b) Let $j, k \in S$ and assume $k$ is accessible from $j$, and state $j$ is recurrent. 
TRUE or FALSE: State $k$ must be recurrent. Explain your answer briefly.

(c) TRUE or FALSE: If every pair of states communicates, the chain must settle down into a stable equilibrium (in the sense we talked about in class where the probability of the chain being in a state converges to a non-zero value). Explain your answer briefly.

(d) The chain has an absorbing state $s_0$. 
TRUE OR FALSE: $\lim_{n \to \infty} P(X_n = s_0) = 1$. Explain your answer briefly.