You can submit this assignment in one of two ways:

1. In class on September 7th.
2. Via Blackboard any time before September 8th at 5pm.

1. Suppose we toss a fair coin until we get exactly two heads. Describe the sample space \( S \).

   Calculate the probability mass function of the random variable describing the number of tosses, i.e., calculate the probability that exactly \( k \) tosses are required for each possible value of \( k \).

2. Prove the union bound inequality, i.e., for events \( A_1, \ldots, A_n \) which are not necessarily disjoint, show that

\[
P \left( \bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} P(A_i).
\]

**Hint:** Define a sequence of events \( B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i \), argue that these events are disjoint. Show that the union of \( B_i \)s is equal to the union of \( A_i \)s, and use the countable additivity property of probability measures.

3. Suppose that \( A \) and \( B \) are independent events. Show that \( A^c \) and \( B^c \) are independent events.

4. Show that if \( P(A) = 0 \) or \( P(A) = 1 \) then \( A \) is independent of every other event. Show that if \( A \) is independent of itself then \( P(A) \) is either 0 or 1.

5. Let \( X \) have CDF \( F \). Find the CDF of \( X^+ = \max\{0, X\} \).

6. A CDF \( F_X \) is stochastically greater than a CDF \( F_Y \) if \( F_X(t) \leq F_Y(t) \) for all \( t \) and \( F_X(t) < F_Y(t) \) for some \( t \). Prove that in this case, if \( X \) has CDF \( F_X \) and \( Y \) has CDF \( F_Y \) then,

\[
P(X > t) \geq P(Y > t) \quad \text{for every } t,
\]

and

\[
P(X > t) > P(Y > t) \quad \text{for some } t.
\]

7. Prove that the function: \( F_X(x) = 1 - \exp(-x) \), for \( x \in (0, \infty) \) is a valid CDF.

8. The uniform distribution on \([0, 2]\) has density:

\[
f_X(x) = \frac{1}{2} \quad \text{for } x \in [0, 2].
\]

Suppose \( X \) has this density, i.e., that \( X \) is a random variable that is uniformly distributed on \([0, 2]\) calculate:

(a) \( P(X = 1) \).

(b) \( P(0.5 \leq X \leq 1.5) \).