HW2: 36-700 (Fall 2016)

You can submit this assignment in one of two ways:

1. In class on September 14th.
2. Via Blackboard any time before September 15th at 5pm.

1. Suppose we generate a random variable $X$ in the following way. First we flip a fair coin. If the coin is heads, take $X$ to have a $U(0,1)$ distribution. If the coin is tails, take $X$ to have a $U(3,4)$ distribution.
   
   (a) Find the mean of $X$.
   
   (b) Find the standard deviation of $X$.

2. For a collection of random variables prove that:

   
   \[
   \text{Variance}\left(\sum_{i=1}^{n}a_iX_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_ia_j\text{Cov}(X_i, X_j).\]

   You can use the following result: for a set of numbers $x_1, \ldots, x_n$,

   \[
   \left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_ix_j.
   \]

3. Show that for a standard normal random variable $X$, the moment generating function is:

   \[
   M_X(t) = \exp(t^2/2).
   \]

**Hint:** You might find the fact that the Gaussian density integrates to 1 useful for your calculation, i.e.

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2) dx = 1.
\]

4. Suppose that $X_1, \ldots, X_n$ are random variables, then we define the sample mean to be:

   \[
   Y = \frac{1}{n} \sum_{i=1}^{n} X_i,
   \]

   and the sample variance to be:

   \[
   S = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - Y)^2.
   \]

   Suppose that $X_1, \ldots, X_n$ are independent and identically distributed with mean $\mathbb{E}[X] = \mu$ and variance $\text{Var}(X) = \sigma^2$. Show that,
(a) \( \mathbb{E}[Y] = \mu. \)
(b) \( \text{Var}[Y] = \frac{\sigma^2}{n}. \)
(c) \( \mathbb{E}[S] = \sigma^2. \)

5. Let \((X, Y)\) have the uniform distribution on \([0, 1] \times [0, 1]\). Find the probability that \(X + Y \geq 1/2.\)

6. Let \((X, Y)\) have the uniform distribution on the triangle with vertices \((0, 0), (0, 1),\) and \((1, 0)\). Find the joint density function of \((X, Y)\).

7. Let \(F\) be a CDF, and \(U\) a random variable uniformly distributed on \([0, 1]\). Then show that the random variable \(Z := F^{-1}(U)\) has CDF \(F\).

Effectively this result lets us draw samples from any distribution whose CDF is known, via samples from the uniform distribution on \([0, 1]\).