1. Suppose that we take a collection of sets $\mathcal{A}$, and a collection of sets $\mathcal{B}$, and define $\mathcal{C}$ as:

$$C = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$ 

Show that the shattering number:

$$s(\mathcal{C}, n) \leq s(\mathcal{A}, n) \times s(\mathcal{B}, n).$$

2. Suppose instead of taking the union of individual sets, we simply collected all sets to define:

$$C = \{A : A \in \mathcal{A} \text{ or } A \in \mathcal{B}\}.$$ 

Show that the shattering number:

$$s(\mathcal{C}, n) \leq s(\mathcal{A}, n) + s(\mathcal{B}, n).$$

3. Suppose that $\theta > 0$, let $X_1, \ldots, X_n \sim \text{Unif}(-\theta, \theta)$:

   (a) Write down the likelihood function.
   (b) Find a minimal sufficient statistic.
   (c) Show that $X_1$ is not a sufficient statistic.

4. Suppose that $X_1, \ldots, X_n \sim N(\mu, \mu^2)$. Compute the likelihood function. Find a minimal sufficient statistic.

5. Suppose that we have $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. We begin with the initial estimator

$$\hat{\theta} = X_1,$$

and take the sufficient statistic $T(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i$.

   (a) Compute the Rao-Blackwellized estimator.
   (b) Compute the risk of $\hat{\theta}$ and compare it to the risk of the Rao-Blackwellized estimator.
   (c) Suppose you instead tried to compute the “estimator” $E[X_1|X_2]$. Show that this is infeasible.
6. For the following distributions, check that they are in an exponential family by appropriately re-writing their density. You can look up the pdfs/pmf online. List the sufficient statistics and canonical parameters.

(a) The Gamma distribution with shape parameter $k$ and scale $\theta$.
(b) The central $\chi^2$ distribution with $k$ degrees of freedom.
(c) The multinomial distribution (treat the number of trials $n$ as fixed).

In each case, also compute the log-partition function $A$ and differentiate it to obtain the expected value of the sufficient statistics.