Practice Exam 1 Questions
36-705

1. Random variables which are exponentially concentrated around their means have many pleasant properties. Show that, if a random variable $X$ satisfies:

$$
P(|X - \mathbb{E}[X]| \geq t) \leq c_1 \exp(-c_2 t^2),$$

for some constants $c_1, c_2 > 0$, then any median of $X$ cannot be too far from $\mathbb{E}[X]$. Concretely, upper bound the distance between any median of $X$ and its mean by some constant that depends only on $c_1, c_2$ above.

2. In this problem we consider an extreme simplification of the standard classification setup: we observe training samples $\{(X_1, y_1), \ldots, (X_n, y_n)\}$ where $X \in \mathbb{R}^d$ and $y \in \{-1, 1\}$ from a distribution $P$. We are interested in evaluating a classifier $f : \mathbb{R}^d \mapsto \{-1, 1\}$. In order to do this we evaluate its empirical risk:

$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(y_i \neq f(X_i)).$$

(a) What is the expected value of the empirical risk?
(b) Provide an exponential concentration inequality for the empirical risk.
(c) Does the exponential concentration inequality you provided depend on any non-trivial property of the distribution $P$?

3. Show that the central limit theorem implies the WLLN, i.e. show that if,

$$\sqrt{n} \left( \frac{Y_n - \mu}{\sigma} \right) \overset{d}{\rightarrow} \mathcal{N}(0, 1),$$

then $Y_n \overset{p}{\rightarrow} \mu$.

4. Suppose $X_1, \ldots, X_n$ are i.i.d with mean $\mu$ and variance $\sigma^2$. Let

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Show that,

$$\sqrt{n} \left( \frac{1}{\hat{\mu}} - \frac{1}{\mu} \right) \overset{d}{\rightarrow} \mathcal{N} \left( 0, \frac{\sigma^2}{\mu^4} \right).$$

Suppose you wanted to construct an approximate confidence interval and did not know $\mu, \sigma$. How would you construct an approximate confidence interval? Justify your answer.

5. Compute the VC dimension of $d$-dimensional hyper-rectangles. Concretely, suppose you think the VC dimension is $v$, you need to show a set of $v$ points that is shattered and then argue that no set of $v + 1$ points can be shattered.