Homework 1: Solutions (The analytical parts)
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1 (a) As hinted, use Bayes’:
\[ P(D|T) = \frac{P(T|D)P(D)}{P(T)} \]  
\( P(T|D) = 0.99 \)  
\( P(D) = 1/50000 \)  
\[ P(T) = P(T|D)P(D) + P(T|D^C)P(D^C) \]  
\[ = 0.99 \times 1/50000 + 0.01 \times 49999/50000 \]

Put all that together, and we get 0.002 or so. What is happening is that since the disease is so uncommon, the false positives dominate, even though the test is correct most of the time in an isolated case, that is \( P(T|D) = P(T^C|D^C) = 0.99 \).

(b) See the code provided in the homework problem statement. You should see numbers that look something like:

TRUE POSITIVES: 27
TRUE NEGATIVES: 989964
FRACTION OF TRUE POSITIVES: 0.002690315

2 (a) We have:
\[ Y = g(U) = U^{1/\alpha} \]  
\[ \implies U = g^{-1}(Y) = Y^{\alpha} \]

This gives us:
\[ p(Y)dy = p_U(Y^{\alpha})\alpha Y^{\alpha-1}dU \]  
\[ = \alpha Y^{\alpha-1}I_{Y \in [0,1]}dU \]

Consult the ancient texts and you will see that this is indeed a Beta distribution with parameters \((\alpha, 1)\). Note that the indicator works out since both \(0^{\alpha} = 0\) and \(1^{\alpha} = 1\).

(b) This can be done in any number of ways: scatterplots, histograms, q-q plots, K-S tests. Anything convincing gets full credit.

(c) Here:
\[ P(Y \leq y) = P(U^{-\alpha} \leq y) \]  
\[ = P(U \geq y^{-\alpha}) \]  
\[ = 1 - P(U \leq y^{-\alpha}) \]  
\[ = 1 - y^{-\alpha}I_{y \in [1,\infty]} \]

Note that the indicator comes from considering which \( y \) will produce \( y^{-\alpha} \in [0, 1] \).
(d) This can be done in any number of ways: scatterplots, histograms, q-q plots, K-S tests. Anything convincing gets full credit.

3 (a) We will integrate out \( v \) to obtain a marginal:

\[ p(x) = \int p(x|v)p(v)dx \]

\[ = \int \left( \frac{\sqrt{v}}{\sqrt{2\pi}} e^{-x^2v/2} \right) \left( \frac{\left( \frac{a}{2} \right)^{a/2} v^{a/2-1} e^{-va/2}}{\Gamma(a/2)} \right) \, dv \]

\[ = \frac{\left( \frac{a}{2} \right)^{a/2}}{\Gamma(a/2)\sqrt{2\pi}} \int v^{a/2-1/2} e^{-v(a/2+x^2/2)} \, dv \]

\[ = \frac{\left( \frac{a}{2} \right)^{a/2}}{\Gamma(a/2)\sqrt{2\pi}(a/2 + x^2/2)^{a/2+1/2}} \]

\[ = \frac{\Gamma(a/2 + 1/2)}{\Gamma(a/2)} \frac{1}{\sqrt{2\pi}} \frac{\left( \frac{a}{2} \right)^{a/2}}{\left( \frac{a}{2} \right)^{a/2+1/2} (1+x^2/a)^{a/2+1/2}} \]

\[ = \frac{\Gamma(a/2 + 1/2)}{\Gamma(a/2)} \frac{1}{\sqrt{\pi}} \frac{1}{(1+x^2/a)^{a/2+1/2}} \]

The last line is the t distribution we seek. We did the following: (1) state the definition of total probability, (2) write out the given densities, (3) factor out everything that does not depend on \( v \), we now recognize that the inside is the kernel (the part that depends on \( v \)) of a gamma distribution with parameters \( a^* = a/2+1/2, b^* = (a/2+x^2/2) \), (4) now results from multiplying and dividing by the appropriate normalizing constant for that gamma kernel, giving us an integral that is unity, leaving us to collected terms in (5) and (6) towards the final obtaining of the quested for quantity.

(b) This can be done in any number of ways: scatterplots, histograms, q-q plots, K-S tests. Anything convincing gets full credit.

(c) Here we do the same dance:

\[ p(y) = \int p(y|\lambda)p(\lambda)d\lambda \]

\[ = \int \frac{e^{-\lambda y} b^a e^{-b\lambda} \lambda^{a-1}}{y! \Gamma(a)} \, d\lambda \]

\[ = \frac{b^a}{y!\Gamma(a)} \int e^{-\lambda(1+b)} \lambda^{y+a-1} \, d\lambda \]

\[ = \frac{\Gamma(y+a)b^a}{y!\Gamma(a)(1+b)^{y+a}} \]

\[ = \frac{\Gamma(y+a) \left( \frac{b}{1+b} \right)^a \left( \frac{1}{1+b} \right)^y}{y!\Gamma(a)} \]

The last line is indeed a negative binomial. With \( p = 1/(1+b) \) and with number of successes \( a \).
(d) This can be done in any number of ways: scatterplots, histograms, q-q plots, K-S tests. Anything convincing gets full credit.

4 (a) Dum de dum:

\[ p(\beta | x, \alpha) \propto p(x | \alpha, \beta) p(\beta) \]

\[ \propto \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x} \frac{b^\alpha}{\Gamma(a)} \beta^{a - 1} e^{-b\beta} \]

\[ \propto \beta^{\alpha + a - 1} e^{-\beta(x + b)} \]

So it is the kernel of another gamma. Note that for a sample, we replace \( \alpha \) with \( n\alpha \) and \( x \) with \( \sum x_i = nx \).

(b) La de da:

\[ p(\theta | x) \propto p(x | \theta) p(\theta) \]

\[ \propto \frac{1}{\theta} I_{x \in [0, \theta]} I_{\theta > a} \frac{\alpha \theta^\alpha}{\theta^{\alpha + 1}} \]

\[ \propto I_{x < \theta} I_{a < \theta} \frac{1}{\theta^{\alpha + 2}} \]

\[ \propto I_{\theta > \max(a, x)} \frac{1}{\theta^{\alpha + 2}} \]

This last line is the kernel of another pareto, with \( \alpha^* = \alpha + 1 \), and \( a^* = \min(a, x) \).

In the case of an iid sample, we would instead have \( a^* = \max(a, \max_i x_i) \).

5 (a) Since \( p(\phi) = I_{\phi \in [0, 1]} \), and \( p(\phi | Y) \propto p(Y | \phi) p(\phi) \), then the MAP density corresponds to the MLE, which is \( \phi_{MAP} = \frac{320}{600} \). You could also observe that the posterior is a beta distribution, and make a statement about the mode of a beta. That is, it is a beta with \( \alpha = 321 \), \( \beta = 281 \), which gives a MAP of \( \frac{\alpha - 1}{\beta + \alpha - 2} = \frac{320}{600} \), the same answer.

(b) You can use \texttt{optim()} here, or just evaluate \texttt{dbeta} on a fine grid. See the R code solutions.

(c) Use \texttt{pieta} here. See the R code solutions.

(d) Use \texttt{pieta} here. See the R code solutions.

6 See the R code section of the solutions.

7 Let \( M(\theta) \) be the minimal \( M \) necessary for an exponential with parameter \( \theta \) to be an acceptable envelope for the normal tail. First, observe that the efficiency of the rejection sampler is given by:

\[ \int \frac{N(x)}{M(\theta) E(\theta, x)} E(\theta, x) dx = \frac{1}{M(\theta)}. \]

So, we want to choose \( \theta \) so \( M(\theta) \) to be as small as possible. Now, we can find \( M \) as follows:

\[ M(\theta) = \left( \min_x \frac{E(\theta, x)}{N(x)} \right)^{-1} \]

The inner quantity is the ratio between the envelope and the normal tail. The smallest such ratio will be the point where we need the largest \( M(\theta) \) so that
the envelope is greater than the tail. Taking the reciprocal of this amount gives
us the amount we must multiply the envelope so that it lies on top of the tail
at that point, and thus over all points (since we already determined that this
is the point with the minimal ratio and thus in need of the greatest amount of
multiplication).

We will find the minimum first, using calculus, let \( \psi = \mathbb{P}(N(-2, 1) > 0) \):

\[
\frac{E(\theta, x)}{N(x)} = \theta \psi \sqrt{2\pi e^{-\theta x + (x+2)^2/2}}
\]

(34)

\[
\frac{d}{dx} \frac{E(\theta, x)}{N(x)} = (-\theta + x + 2) \theta \psi \sqrt{2\pi e^{-\theta x + (x+2)^2/2}}
\]

(35)

We can clearly see that the minimum is at \( x^* = \theta - 2 \). Now, for \( \theta \geq 2 \), the
multiplier is equal to:

\[
M(\theta) = 1 \frac{\psi}{\theta \sqrt{2\pi e^{\theta^2/2}}} \sqrt{2\pi e^{\theta^2/2-2\theta}}
\]

(36)

\[
= 1 \frac{\psi}{\theta \sqrt{2\pi}} e^{\theta^2/2-2\theta}
\]

(37)

For \( \theta < 2 \), we can see, by examining the derivative \( \frac{d}{dx} \frac{E(\theta, x)}{N(x)} \), that \( x^* = 0 \) will
minimize the ratio, and so the multiplier is equal to:

\[
M(\theta) = e^{2 \frac{\psi}{\theta \sqrt{2\pi}}}
\]

(38)

Since as \( \theta \) decreases, this increases, we can discount any \( \theta < 2 \). Now, for \( \theta \geq 2 \),
we take the derivative:

\[
M(\theta) = 1 \frac{\psi}{\theta \sqrt{2\pi e^{\theta^2/2}}} \sqrt{2\pi e^{\theta^2/2-2\theta}}
\]

(39)

\[
\frac{d}{d\theta} M(\theta) = -1 \frac{\psi}{\theta \sqrt{2\pi e^{\theta^2/2}}} e^{\theta^2/2-2\theta} + \frac{\theta - 2}{\theta \psi \sqrt{2\pi}} e^{\theta^2/2-2\theta}
\]

(40)

Setting equal to zero and clearing many annoying constants and exponentials
we arrive at:

\[
0 = -1 + \theta^2 - 2\theta
\]

(41)

Which is true for \( \theta = 1 \pm \sqrt{2} \), which gives us only one valid solution: \( \theta = 1 + \sqrt{2} \).

Run the R code solutions to see a graphical display of why this is the best
\( \theta \).