Chapter 8, Problem 1. $Y = n\hat{F}_n(x) \sim \text{Binomial}(n, p)$ where $p = \mathbb{P}(X < x) = F(x)$. So $\mathbb{E}(\hat{F}(x)) = \mathbb{E}(Y/n) = F(x)$ and $\text{Var}(\hat{F}(x)) = n^{-2}\text{Var}(Y) = p(1-p)/n = F(x)(1-F(x))/n$. The MSE is $\text{bias}^2 + \text{Var} = \mathbb{V} = F(x)(1-F(x))/n$. Hence $\text{mse} \to 0$ which implies that $\hat{F}(x) \xrightarrow{a} F(x)$ which implies that $\hat{F}(x) \xrightarrow{P} (x)$.

Chapter 8, Problem 2. The quantity of interest is $\delta = p - q = \int xdF_1(x) - \int xdF_2(x)$ and the plug-in estimator is $\hat{\delta} = \int xd\hat{F}_1(x)-\int xd\hat{F}_2(x) = n^{-1}\sum_i X_i - m^{-1}\sum_i Y_i = \hat{p} - \hat{q}$. The variance of the estimator is $\mathbb{V}(\hat{\delta}) = \mathbb{V}(\hat{p} - \hat{q}) = \mathbb{V}(\hat{p}) + \mathbb{V}(\hat{q}) = p(1-p)/n + q(1-q)/m$ and the estimated standard error is

$$\hat{se} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{\hat{q}(1-\hat{q})}{m}}.$$ 

Since $z_{0.05} = 1.64$, an approximate 90 per cent interval is

$$\hat{p} - \hat{q} \pm 1.64 \sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{\hat{q}(1-\hat{q})}{m}}.$$ 

Chapter 8, Problem 3. We can write

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

where $Y_i \sim \text{Bernoulli}(p)$ with $\mathbb{E}(Y_i) = p = F(x)$ and $\text{Var}(Y_i) = p(1-p) = F(x)(1-F(x))$. Hence,

$$\sqrt{n}(\hat{F}(x) - F(x)) \xrightarrow{d} N(0,1).$$
6. 

\[ \mathbb{E}(\hat{\lambda}) = \mathbb{E}(X_1) = \lambda \]

and hence \( \text{bias} = 0 \).

\[ \mathbb{V}(\hat{\lambda}) = \frac{\mathbb{V}(X_1)}{n} = \frac{\lambda}{n} \]

and \( \text{se} \sqrt{\lambda/n} \)

7. The CDF \( G \) of \( \hat{\theta} \) is

\[
G(\hat{\theta}) = \mathbb{P}(\hat{\theta} \leq \hat{\theta})
= \mathbb{P}(\max\{X_1, \ldots, X_n\} \leq \hat{\theta})
= \prod_{i=1}^{n} \mathbb{P}(X_i \leq \hat{\theta})
= F_\theta(\hat{\theta})^n
= \left( \frac{\hat{\theta}}{\theta} \right)^n.
\]

The density is therefore

\[
g(\hat{\theta}) = \left( \frac{n}{\theta} \right) \left( \frac{\hat{\theta}}{\theta} \right)^{n-1}.
\]

Thus,

\[
\mathbb{E}_\theta(\hat{\theta}) = \int_0^\theta \hat{\theta} g(\hat{\theta}) d\hat{\theta} = \frac{n\theta}{n+1}
\]

and

\[
\text{bias} = \frac{n\theta}{n+1} - \theta = -\frac{\theta}{n+1}.
\]

Also,

\[
\mathbb{E}_\theta(\hat{\theta}^2) = \int_0^\theta \hat{\theta}^2 g(\hat{\theta}) d\hat{\theta} = \frac{n\theta^2}{n+2}
\]
and so
\[ V_\theta(\hat{\theta}) = \frac{n\theta^2}{n+2} - \left( \frac{n\theta}{n+1} \right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2}. \]

The mse is
\[ \text{bias}^2 + \text{var} = \left( \frac{\theta}{n+1} \right)^2 \frac{n\theta^2}{(n+2)(n+1)^2} = \frac{2\theta^2}{(n+1)(n+2)}. \]

8. Recall that \( \mathbb{E}(X_i) = \theta/2 \), \( \mathbb{V}(X_i) = \theta^2/12 \). So
\[ \mathbb{E}_\theta(2\bar{X}) = 2\mathbb{E}_\theta(\bar{X}) = \frac{2\theta}{2} = \theta \]
and hence bias = 0. Now
\[ V_\theta(2\bar{X}) = 4V_\theta(\bar{X}) = \frac{4\sigma^2}{n} = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}. \]
Since this estimator is unbiased,
\[ \text{mse} = V_\theta(\hat{\theta}) = \frac{\theta^2}{3n}. \]

9. \( \mu = \mathbb{E}(X_i) = 1/2 \) and \( \sigma^2 = \mathbb{V}(X_i) = 1/12 \). By the CLT,
\[ \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \sqrt{12n} \left( \bar{X} - \frac{1}{2} \right) \sim N(0,1). \]
Now \( Y = g(\bar{X}) \) where \( g(s) = s^2 \). And \( g'(s) = 2s \) and \( g'(\mu) = g'(1/2) = 2(1/2) = 1 \). From the delta method,
\[ \frac{\sqrt{n}(Y - g(\mu))}{|g'(\mu)|\sigma} = \sqrt{12n} \left( \bar{X} - \frac{1}{4} \right) \sim N(0,1). \]

10. \( Y_n = g(\bar{X}_1, \bar{X}_2) \) where \( g(s_1, s_2) = s_1/s_2 \). By the central limit theorem,
\[ \sqrt{n} \left( \frac{\bar{X}_1 - \mu_1}{\bar{X}_2 - \mu_2} \right) \sim N(0, \Sigma). \]
Now
\[ \nabla g(s) = \left( \frac{\partial g}{\partial s_1}, \frac{\partial g}{\partial s_2} \right) = \left( \frac{1}{s_2}, -\frac{s_1}{s_2^2} \right) \]

and so
\[ \nabla^T \Sigma \nabla = \begin{pmatrix} 1 - \frac{\mu_1}{\mu_2} \\ \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_2} \\ -\frac{\mu_1}{\mu_2} \end{pmatrix} = \frac{\sigma_{11}\mu_2^2 - 2\mu_2\mu_1\sigma_{12} + \mu_1^2\sigma_{22}}{\mu_2^4} \]

Therefore,
\[ \sqrt{n(X_1X_2 - \mu_1\mu_2)} \sim N\left(0, \frac{\sigma_{11}\mu_2^2 - 2\mu_2\mu_1\sigma_{12} + \mu_1^2\sigma_{22}}{\mu_2^4} \right). \]