Some New Three Level Designs for the Study of Quantitative Variables

G. E. P. BOX AND D. W. BEHNKEN

University of Wisconsin and the American Cyanamid Company

A class of incomplete three level factorial designs useful for estimating the coefficients in a second degree graduating polynomial are described. The designs either meet, or approximately meet, the criterion of rotatability and for the most part can be orthogonally blocked. A fully worked example is included.

1.0. INTRODUCTION

A symmetrical factorial design is an experimental arrangement in which a small integral number p of levels is chosen for each of k factors (i.e. variables) and all p^k combinations of these levels are run. Classes of these designs which have proved to be of particular interest are those in which two levels or three levels are used for each of the k variables. These are called respectively 2^k and 3^k factorials. If not all the factorial combinations are employed but merely a selected subset, we call the design an incomplete factorial. Any factorial or incomplete factorial we call a factorial-type design.

A class of incomplete factorials of considerable interest are the fractional factorials of D. J. Finney [1] [2]. In these arrangements certain finite group properties are employed to select a $(1/p)^f$ fraction of the complete design which then requires only p^{k-f} combinations of levels and may be called a p^{k-f} factorial. A useful and different class of incomplete factorials in which the selected subset is not restricted to be a $(1/p)^f$ fraction is due to Plackett and Burman [3].

An infinite choice exists for the levels of quantitative variables such as temperature. In developing designs specifically for quantitative variables, there is therefore no essential need to restrict experimental conditions to combinations of a few basic levels of the component factors. Many useful designs have indeed been devised for the study of quantitative variables which do not employ the factorial principle [4] [5]. In spite of this, cases are not uncommon where even though the factors are all quantitative, convenience requires the use of only a few levels for each.

In this paper we discuss a particular class of three-level incomplete factorials specifically selected for the study of quantitative variables. The class of designs is not included among the types of incomplete factorials already discussed but nevertheless appears to be of considerable practical importance.

2.0. INCOMPLETE FACTORIALS FOR QUANTITATIVE VARIABLES

When a design involving N runs is employed to separately estimate L constants we may define the ratio $R = N/L$ as the *redundancy factor* for the design. This factor is necessarily not less than unity.

Suppose in what follows that the functional relationship between the response of interest and the levels of the k quantitative experimental variables may be graduated by a general polynomial of degree d in the levels of the variables. A design suitable for separately estimating the $(k + d)!/k!\,d!$ constants of such a polynomial is called a *design of order d*. The highest degree of polynomial that may be fitted to the observations from a p-level factorial is $p - 1$. Consequently when regarded as a design for the fitting of a general polynomial the p^k factorial is a design of order $p - 1$. The redundancy factor for such a design is therefore $p^k k!(p - 1)!/(k + p - 1)!$. When calculated in this way the redundancy factors for the complete factorials are usually large. For example, regarded as a first order design, the two-level factorial in five factors requires $2^5 = 32$ runs to estimate the 6 constants of the first degree polynomial. It therefore has a redundancy factor of $32/6 = 5.3$. Similarly, regarded as a second order design the three-level factorial in five factors requires $3^5 = 243$ runs to estimate the 21 constants of the second degree polynomial. It therefore has a redundancy factor of $243/21 = 11.6$.

In situations in which the experimental error variance is not so large as to require large numbers of observations to obtain necessary precision, designs having small redundacy factors are desirable. Small redundacy factors may sometimes be obtained by using incomplete rather than complete factorial designs. For example, if $k = 3, 7, 11, 15, \cdots, 4i - 1$ the two-level arrangements of Plackett and Burman provide first order designs requiring respectively only $4, 8, 12, 16, \cdots, 4i$ runs, where i is a positive integer. They are thus first order two-level designs of redundancy unity. Designs having this minimal redundancy are seldom employed in practice because they provide no residual degrees of freedom and so do not allow the possibility of partially checking [6] [7] the adequacy of the assumed form of model. Other incomplete two-level factorial designs are available however having low redundancy factors of two or less which do not suffer from this deficiency.

For the presently available three-level factorials the situation is less satisfactory than for the two-level designs. For example, the various one-ninth replicates of the 3^5 factorials all seem to lead to undesirable correlation or confounding of estimates of the coefficients and although a one-third replicate of the 3^5 factorial may be employed as a second order design it has a redundancy factor of $81/21 = 3.9$ which is somewhat high.

In developing the present class of designs we do not use the group properties exploited by Finney; rather we set out directly to select part of the 3^k factorial which allows efficient estimation of a second degree graduating polynomial. Specifically, we have where possible set out to generate second order rotatable designs. Arguments in favor of such a choice have been presented elsewhere [5]. Suppose we code the levels in standardized units so that the 3 values taken by each of the variables $x_1, x_2, \cdots x_k$ are -1, 0, and 1 and suppose also that the second degree graduating polynomial fitted by the method of least squares is

$$\hat{y} = b_0 + \sum_{i=1}^{k} b_i x_i + \sum_{i=1}^{k} \sum_{j=i}^{k} b_{ij} x_i x_j .$$

A second order rotatable design is such that the variance of $\hat{y}$ is constant for all points equidistant from the center of the design—that is, for all points for which $\rho = (\sum_i x_i^2)^{\frac{1}{2}}$ is constant. Among the class of rotatable designs we select those for which the variance of $\hat{y}$, regarded as a function of ρ, is reasonably constant in the region of the k-space covered by the design. The requirement of rotatability is introduced to ensure a symmetric generation of information in the space of the variables defined and scaled in a manner currently thought most appropriate by the experimenter. For a design to be useful it need not have the property of rotatability exactly. For certain values of k, it turns out that within the class of designs we consider, rotatability can be achieved exactly; in other cases, exact rotatability is not possible and here, as described more fully in Appendix A, we relax the requirement to some extent. All the designs we discuss possess a high degree of orthogonality; in fact, only the constant term b_0 and the quadratic estimates b_{ii} are correlated* one with another.

The requirement of rotatability or near-rotatability imposes certain restrictions [5] on the moments of the design. In Appendix A it is shown that when these restrictions are applied to variables which can take only the values -1, 0, and 1 certain simple combinatorial requirements emerge and that these requirements can be satisfied by combining two-level factorial designs and incomplete block designs in a particular manner exemplified in the next section.

The existence of the class of designs discussed here was suggested by the discovery in another connection [8] of a three-level rotatable design in seven variables which required only 56 points plus added points at the origin thus providing highly efficient estimates of the 36 constants in the polynomial of second degree. Further investigation led to the development of the present class of three-level designs utilizing the properties of incomplete blocks.

3.0. Method for Generating the Designs

The designs are formed by combining two-level factorial designs with incomplete block designs in a particular manner. This is best illustrated by an example. In Table 1 is shown a balanced incomplete block design for testing $k = 4$ varieties in $b = 6$ blocks of size $s = 2$.

Table 1

A balanced incomplete block design for four varieties in six blocks.

$k = 4$ varieties

$b = 6$ blocks		x_1	x_2	x_3	x_4
	1	*	*		
	2			*	*
	3	*			*
	4		*	*	
	5		*		*
	6	*		*	

* Designs for which there is no correlation between either all or a subset of the quadratic coefficients can be obtained but they do not seem to possess any particular advantage [5] so far as estimating the response is concerned.

TABLE 2

A 2^2 factorial design.

$$\begin{array}{cc} x_i & x_j \\ \end{array}$$
$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

If this design were being used in the usual way, varieties 1 and 2 denoted by x_1 and x_2 would be tested in the first block, varieties 3 and 4 in the second, and so on.

A basis for a three-level design in four variables is obtained by combining this incomplete block design with the 2^2 factorial of Table 2. The two asterisks in every row of the incomplete block design are replaced by the $s = 2$ columns of the two-level 2^2 design. Wherever an asterisk does *not* appear a column of zeros is inserted. The design is completed by the addition of a number of center points (0, 0, 0, 0), about three being desirable with this arrangement. The resulting design is shown in Table 3. As explained later, this design can in fact be run in three orthogonal blocks. These are indicated by dotted lines in the table.

The design obtained is a rotatable second order design suitable for studying four variables in 27 trials and is capable of being blocked in three sets of nine trials. It is shown in Appendix B that this particular design is in fact a rotation of the corresponding central composite rotatable design [5] in four variables. It is however not generally true that the present class of designs can be generated from the central composite designs by rotation.

TABLE 3

An incomplete 3^4 factorial in three blocks of nine experimental runs.

x_1	x_2	x_3	x_4	
−1	−1	0	0	
1	−1	0	0	
−1	1	0	0	
1	1	0	0	
0	0	−1	−1	Block 1
0	0	1	−1	
0	0	−1	1	
0	0	1	1	
0	0	0	0	
−1	0	0	−1	
1	0	0	−1	
−1	0	0	1	
1	0	0	1	
0	−1	−1	0	Block 2
0	1	−1	0	
0	−1	1	0	
0	1	1	0	
0	0	0	0	
0	−1	0	−1	
0	1	0	−1	
0	−1	0	1	
0	1	0	1	
−1	0	−1	0	Block 3
1	0	−1	0	
−1	0	1	0	
1	0	1	0	
0	0	0	0	

In Table 4 a number of designs of the class under study are given suitable for investigating 3, 4, 5, 6, 7, 9, 10, 11, 12, and 16 variables. In this table unless otherwise indicated the symbol $(\pm 1, \pm 1, \cdots, \pm 1)$ means that all combinations of plus and minus levels are to be run. Whenever a fractional factorial is available which does not confound main effects and two factor interactions one with another, it may be used instead of the full factorial. For example, in design No. 8, s is equal to five and as indicated in the table rather than using a full 2^5 factorial we can achieve the desired result with a half-replicate.

Three members of the class of designs have been generated by other methods and have appeared elsewhere. Design No. 1 was first described by DeBaun [9], [10] and design No. 2 by Gardiner, Grandage and Hader [11]. The general method of Bose and Draper rederived design No. 2 in [12] and produced the points in designs No. 1 and No. 3 as identifiable subsets of rotatable designs in [13] and [12] respectively.

4.0. Blocking the Designs

Where insufficient homogeneous experimental material is available for all the experimental runs it becomes desirable to run them in blocks. Where possible it is desirable to achieve *orthogonal* blocking, that is to arrange that the block constrasts are uncorrelated with all the estimates of the coefficients in the polynomial. When this can be achieved the analysis may be carried out almost as if block differences did not exist. The only modification necessary is that in the analysis of variance table the sum of squares associated with block differences must be substracted from the residual sum of squares. On the assumption that the model is adequate, the residual sum of squares so adjusted may then be used to estimate the within-block variance and hence the standard errors of the coefficients.

The requirements for orthogonal blocking of second order designs have been given elsewhere [5]. Applying these results to the present problem, it is easy to see that:

(1) Where "replicate sets" can be found in the generating incomplete block design these provide a basis for orthogonal blocking. These replicate sets are subgroups within which each variety is tested the same number of times.

(2) Where the component factorial designs can be divided into blocks which only confound interactions of more than two factors these can provide a basis for orthogonal blocking.

An illustration of the first method of blocking has already been given in the example of Section 3.0. In Table 4 dotted lines indicate the appropriate divisions into replicate sets. Using these divisions design No. 2 can be split into three blocks, design No. 3 into two blocks, design No. 6 into five blocks and design No. 10 into six blocks. In these and other blocking schemes discussed below, the center points *must* be distributed equally among blocks to retain orthogonality.

The second method may be illustrated with design No. 4 for which the first method cannot be employed. The basis for the design consists of 48 trials gen-

TABLE 4

Some useful three-level designs

Design Number	Number of Factors(k)	Design Matrix	No. of Points	Blocking and Association Schemes
1	3	$\begin{bmatrix} \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 \\ 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 \end{bmatrix}$	12 3 $N = 15$	No orthogonal blocking BIB (one associate class)
2	4	$\begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & 0 \\ \hline \pm 1 & 0 & 0 & \pm 1 \\ 0 & \pm 1 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \pm 1 & 0 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	8 1 8 1 8 1 $N = 27$	3 blocks of 9 BIB (one associate class)
3	5	$\begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \pm 1 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	20 3 20 3 $N = 46$	2 blocks of 23 BIB (one associate class)

4	6	$\begin{bmatrix} \pm1 & \pm1 & 0 & \pm1 & 0 & 0 \\ 0 & \pm1 & \pm1 & 0 & \pm1 & 0 \\ 0 & 0 & \pm1 & \pm1 & 0 & \pm1 \\ \pm1 & 0 & 0 & \pm1 & \pm1 & 0 \\ 0 & \pm1 & 0 & 0 & \pm1 & \pm1 \\ \pm1 & 0 & \pm1 & 0 & 0 & \pm1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	48 6 $N = 54$	2 blocks of 27. First Associates: (1, 4); (2, 5); (3, 6).
5	7	$\begin{bmatrix} 0 & 0 & 0 & \pm1 & \pm1 & \pm1 & 0 \\ \pm1 & 0 & 0 & 0 & 0 & \pm1 & \pm1 \\ 0 & \pm1 & 0 & 0 & \pm1 & 0 & \pm1 \\ \pm1 & \pm1 & 0 & \pm1 & 0 & 0 & 0 \\ 0 & 0 & \pm1 & \pm1 & 0 & 0 & \pm1 \\ \pm1 & 0 & \pm1 & 0 & \pm1 & 0 & 0 \\ 0 & \pm1 & \pm1 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	56 6 $N = 62$	2 blocks of 31. BIB (one associate class).
6	9	$\left[\begin{array}{ccccccccc} \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 \\ 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \pm1 & \pm1 & \pm1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm1 & \pm1 & \pm1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & \pm1 & \pm1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \pm1 & 0 & 0 & 0 & \pm1 & 0 & 0 & 0 & \pm1 \\ 0 & 0 & \pm1 & \pm1 & 0 & 0 & 0 & \pm1 & 0 \\ 0 & \pm1 & 0 & 0 & 0 & \pm1 & \pm1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \pm1 & 0 & 0 & 0 & 0 & \pm1 & 0 & \pm1 & 0 \\ 0 & \pm1 & 0 & \pm1 & 0 & 0 & 0 & 0 & \pm1 \\ 0 & 0 & \pm1 & 0 & \pm1 & 0 & \pm1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 \\ 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 & \pm1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$	24 2 24 2 24 2 24 2 24 2 $N = 130$	(a) 5 blocks of 26. (b) 10 blocks of 13. First Associates: (1, 4); (1, 7); (4, 7); (2, 5); (2, 8); (5, 8); (3, 6); (3, 9); (6, 9).

TABLE 4—*Continued*

Design Number	Number of Factors(k)	Design Matrix	No. of Points	Blocking and Association Schemes
7	10	$\begin{bmatrix} 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 \\ \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 & 0 & \pm 1 \\ 0 & \pm 1 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 \\ 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & 0 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & \pm 1 & 0 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	160 10 $N = 170$	2 blocks of 85. Second Associates: (1, 8); (1, 9); (1, 10); (2, 6); (2, 7); (2, 10); (3, 5); (3, 7); (3, 9); (4, 5); (4, 6); (4, 8); (5, 10); (6, 9); (7, 8).
8	11	$\begin{bmatrix} 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	176 12 $N = 188$	Use 2^{5-1} fractionated on $x_1x_2x_3x_4x_5$. No orthogonal blocking. BIB (one associate class)
9	12	$\begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 & 0 \\ 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 & \pm 1 \\ \pm 1 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	192 12 $N = 204$	2 blocks of 102. First Associates: (1, 7); (2, 8); (3, 9); (4, 10); (5, 11); (6, 12).

10 16

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\pm1 & \pm1 & 0 & 0 & 0 & \pm1 & 0 & 0 & \pm1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & \pm1 & 0 & 0 & 0 & \pm1 & 0 & 0 & \pm1 \\
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0 & \pm1 & 0 & 0 & \pm1 & \pm1 & 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & 0 & 0 & 0 \\
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\pm1 & 0 & \pm1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & 0 & \pm1 & 0 \\
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0 & 0 & 0 & 0 & 0 & \pm1 & 0 & \pm1 & 0 & \pm1 & 0 & \pm1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\hline
N = 396
\end{array}
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(a) 6 blocks of 66.
(b) 12 blocks of 33.

First Associates:
(1, 5); (1, 9); (1, 13);
(5, 9); (5, 13); (9, 13);
(2, 6); (2, 10); (2, 14);
(6, 10); (6, 14); (10, 14);
(3, 7); (3, 11); (3, 15);
(7, 11); (7, 15); (11, 15);
(4, 8); (4, 12); (4, 16);
(8, 12); (8, 16); (12, 16).

erated from six 2^3 factorial designs. If we were running a single 2^3 factorial design, it could be performed in two sets of four trials, confounding the three-factor interaction with blocks. Trials with levels (1, 1, 1), (1, −1, −1), (−1, −1, 1), (−1, 1, −1) would be included in one set (called the positive set) and trials with levels (−1, −1, −1), (−1, 1, 1), (1, −1, 1), (1, 1, −1) in the other (called the negative set). The complete group of 48 trials can be split into two orthogonal blocks of 24 by allocating one set (either positive or negative) from each of the 2^3 factorial designs to one block, and the remainder to the other.

This method is used where the block size $s > 2$ and employed for designs 4, 5, 6, 7, 9, and 10 in Table 4. In designs 7, 9, and 10 the basic factorial is a 2^4 design. This is split into two sets in such a way as to confound the four factor interaction, that is to say trials with levels whose product is positive are allocated to one group, and the remainder to the other.

In some cases, both methods may be used simultaneously. Thus in design 6 the basic incomplete block design contains five "replicates" indicated by the dotted lines in the table, providing a basis for generating five blocks of 24 runs. Each one of these blocks may now be split into two by allocating the positive sets of the component factorials to one block and the negative sets to the other. We obtain finally an arrangement for generating ten blocks of twelve runs. A similar procedure may be applied in blocking design No. 10.

While orthogonal blocking is desirable, since it minimizes the variance of the estimates of the regression coefficients, non-orthogonal blocking schemes may be employed without an excessive loss of precision when smaller block sizes than those given above are required. Such schemes will not be discussed in the present communication.

5.0. Inclusion of Center Points

In addition to the runs generated directly from the 2^s factorial design it is also necessary to include n_o center points in order to avoid singularity in the moment matrix. The number of center points affects the variance profile, that is, the variance of $\hat{y}$ regarded as a function of the distance $\rho = \sqrt{\Sigma x_i^2}$ from the center of the design. The exact number of center points is not critical. The numbers given in the table are chosen so that the variance profile will be reasonably uniform over the region of the experimental design and so that an even number of center points appear in each block. The variance profiles resulting from the designs here considered are shown in Figure 1 of Appendix A.

6.0. Analysis for the Designs

In Tables 5a, 5b, and 5c, formulae and constants are given which are needed for the analysis of the designs of Table 4. The notation is explained below.

6.1. *Calculation of the estimates.*

In order to calculate the estimates b_0, b_i, b_{ij}, b_{ii}, it is first necessary to write out the levels for each of the variables in the design and then to add further columns corresponding to x_1^2, x_2^2, $\cdots$, x_k^2, x_1x_2, x_1x_3, $\cdots$, $x_{k-1}x_k$. This is done in Table 6 for design No. 2 where a set of typical data is also shown

TABLE 5a

Estimates of the regression coefficients and their variances.

$$b_0 = \bar{y}_0$$

$$b_i = A\{iy\}$$

$$b_{ii} = B\{iiy\} + C_1 \sum_{j}^{n_1} \{jjy\} + C_2 \sum_{l}^{n_2} \{lly\} - (\bar{y}_0/s)$$

where $\sum_{j}^{n_1}$ and $\sum_{l}^{n_2}$ refer to summation over first and second associates of i.

$b_{ij} = D_1\{ijy\}$	i, j first associates.
$b_{ij} = D_2\{ijy\}$	i, j, second associates.
$V(b_0) = \frac{1}{n_0}\sigma^2$	
$V(b_i) = A\sigma^2$	
$V(b_{ii}) = [B + 1/s^2n_0]\sigma^2$	
$V(b_{ij}) = D_1\sigma^2$	i, j first associates.
$= D_2\sigma^2$	i, j second associates.
$\mathrm{Cov}\,(b_0b_{ii}) = -\frac{1}{s^2n_0}\sigma^2$	
$\mathrm{Cov}\,(b_{ii}b_{jj}) = \left[C_1 + \frac{1}{s^2n_0}\right]\sigma^2$,	i, j first associates.
$= \left[C_2 + \frac{1}{s^2n_0}\right]\sigma^2$,	i, j second associates.

NOTE: For BIB designs, all i, j are considered first associates and $C_2 = D_2 = 0$. The constants A, B, etc. for the various designs are given in Table 5c.

for illustration. The sum of products of the entries in the columns with the observations y are next calculated. In addition $\bar{y}_0$ the average value of the observations made at the center points is shown. The calculated quantities are next substituted in the formulae given in Table 5a to provide the required estimates using the constants of Table 5c.

The following notation is employed:

$$\{iy\} = \sum_{u=1}^{N} x_{iu}y_u, \quad \{iiy\} = \sum_{u=1}^{N} x_{iu}^2y_u, \quad \{ijy\} = \sum_{u=1}^{N} x_{iu}x_{ju}y_u.$$

The grand total can be regarded as the sum of products between y and a dummy variable x_0 which always takes the value 1 so that

$$\{0y\} = \sum_{u=1}^{N} y_u.$$

TABLE 5b

Formulae for the analysis of variance.

Correction due to the mean:	$\{0y\}^2/N$
Sum of squares due to linear terms:	$A \sum_{i=1}^{k} \{iy\}^2$
Sum of squares due to second degree terms:	
(a) Due to interaction terms:	$D_1 \sum_{i<j}^{n_1} \{ijy\}^2 + D_2 \sum_{i<j}^{n_2} \{ijy\}^2$
(b) Due to quadratic terms:	$b_0\{0y\} + \sum_{i=1}^{k} b_{ii}\{iiy\} - \{0y\}^2/N$
Total sum of squares after correction for the mean:	$\sum_{u=1}^{N} y_u^2 - \{0y\}^2/N$

In the present example the sums of products are:

$$\bar{y}_0 = 90.6; \quad \{0y\} = 2319.4;$$

$$\{1y\} = 23.2; \quad \{2y\} = -23.5; \quad \{3y\} = 13.6; \quad \{4y\} = -44.1;$$

$$\{11y\} = 1033.6; \quad \{22y\} = 1010.3; \quad \{33y\} = 1027.0; \quad \{44y\} = 1024.3;$$

$$\{12y\} = -6.7; \quad \{13y\} = -15.3; \quad \{14y\} = 3.8; \quad \{23y\} = -6.7;$$

$$\{24y\} = -10.5; \quad \{34y\} = -17.0;$$

TABLE 5c

Constants for the designs of Table 4.

Design	A	B	C_1	C_2	D_1	D_2	s	Center Points n_0	Redundancy Factor	Non-Sphericity Index I
1	1/8	1/4	−1/16	0	1/4	0	2	3	1.2	0.38
2	1/12	1/8	−1/48	0	1/4	0	2	3	1.6	0
3	1/16	1/12	−1/96	0	1/4	0	2	6	1.9	0.17
4	1/24	17/216	−10/216	−1/216	1/16	1/8	3	6	1.7	0.23
5	1/24	1/16	−1/144	0	1/8	0	3	6	1.6	0
6	1/40	1/30	−1/120	−1/720	1/16	1/8	3	10	2.2	0.25
7	1/64	17/512	1/512	−7/512	1/16	1/32	4	10	2.4	0.09
8	1/80	1/48	−1/600	0	1/32	0	5	12	2.3	0.06
9	1/64	23/1024	−9/1024	−1/1024	1/32	1/16	4	12	2.1	0.16
10	1/96	41/3072	−7/3072	−1/3072	1/32	1/16	4	12	2.5	0.18

TABLE 6

Sample calculation for the four-factor design (No. 2).

x_1	x_2	x_3	x_4	x_1^2	x_2^2	x_3^2	x_4^2	x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4	y
−1	−1	0	0	1	1	0	0	1	0	0	0	0	0	84.7
1	−1	0	0	1	1	0	0	−1	0	0	0	0	0	93.3
−1	1	0	0	1	1	0	0	−1	0	0	0	0	0	84.2
1	1	0	0	1	1	0	0	1	0	0	0	0	0	86.1
0	0	−1	−1	0	0	1	1	0	0	0	0	0	1	85.7
0	0	1	−1	0	0	1	1	0	0	0	0	0	−1	96.4
0	0	−1	1	0	0	1	1	0	0	0	0	0	−1	88.1
0	0	1	1	0	0	1	1	0	0	0	0	0	1	81.8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	93.8
−1	0	0	−1	1	0	0	1	0	0	1	0	0	0	89.4
1	0	0	−1	1	0	0	1	0	0	−1	0	0	0	88.7
−1	0	0	1	1	0	0	1	0	0	−1	0	0	0	77.8
1	0	0	1	1	0	0	1	0	0	1	0	0	0	80.9
0	−1	−1	0	0	1	1	0	0	0	0	1	0	0	80.9
0	1	−1	0	0	1	1	0	0	0	0	−1	0	0	79.8
0	−1	1	0	0	1	1	0	0	0	0	−1	0	0	86.8
0	1	1	0	0	1	1	0	0	0	0	1	0	0	79.0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	87.3
0	−1	0	−1	0	1	0	1	0	0	0	0	1	0	86.1
0	1	0	−1	0	1	0	1	0	0	0	0	−1	0	87.9
0	−1	0	1	0	1	0	1	0	0	0	0	−1	0	85.1
0	1	0	1	0	1	0	1	0	0	0	0	1	0	76.4
−1	0	−1	0	1	0	1	0	0	1	0	0	0	0	79.7
1	0	−1	0	1	0	1	0	0	−1	0	0	0	0	92.5
−1	0	1	0	1	0	1	0	0	−1	0	0	0	0	89.4
1	0	1	0	1	0	1	0	0	1	0	0	0	0	86.9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	90.7

and from Table 5c for design No. 2 we have $A = 1/12$, $B = 1/8$, $C_1 = -1/48$, $D_1 = 1/4$, $s = 2$, $n_0 = 3$ whence, using the formulae of Table 5a,

$$
\begin{array}{llll}
b_0 = 90.6 & b_1 = 1.93 & b_{11} = -1.42 & b_{12} = -1.68 \\
 & b_2 = -1.96 & b_{22} = -4.33 & b_{13} = -3.83 \\
 & b_3 = 1.13 & b_{33} = -2.24 & b_{14} = 0.95 \\
 & b_4 = -3.68 & b_{44} = -2.58 & b_{23} = -1.68 \\
 & & & b_{24} = -2.63 \\
 & & & b_{34} = -4.25.
\end{array}
$$

For example,

$$b_1 = \frac{1}{12}(23.2) = 1.930$$

$$b_{11} = \frac{1}{8}(1033.6) - \frac{1}{48}(4095.2) - \frac{90.6}{2} = -1.416$$

$$b_{12} = \frac{1}{4}(-6.7) = -1.675$$

6.2. *The Analysis of Variance.*

The analysis of the variance table is readily calculated using the relations of Table 5b as follows.

Analysis of Variance Table

	s.s.	d.f.	m.s.
Due to linear terms	268.36	4	67.09
Due to second order terms	294.92	10	29.49
Residual	126.71	12	10.56
Total after eliminating the mean	689.99	26	

The observations recorded at the center point were 93.8, 87.3, and 90.7. Had there been no blocking of the design (that is if the runs had been made entirely in random order) these observations at the center point would have provided two degrees of freedom for estimating the error variance. Their sum of squares for deviations from their mean would have been 21.16 and the residual sum of squares could have been split into two parts, as follows

		s.s.	d.f.	m.s.
Residual	Replicated center points	21.16	2	10.58
	Remainder	105.57	10	10.56
		126.71	12	

to provide a basis for a possible test of goodness of fit for the model.

In this particular example, since the error sum of squares would have only two degrees of freedom, such a test would of course be very insensitive and provide no more than an indication that the remainder sum of squares was or was not of the right order of magnitude. Our main object here is to illustrate general principles.

6.3. *Elimination of Block Effects.*

The design illustrated was actually carried out in three blocks of nine observations. Since the blocking is orthogonal the elimination of blocks will only affect the residual sum of squares. The block means $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$ are respectively 749.1/9, 750.6/9, 774.7/9 and the sum of squares associated with blocks is

$$\frac{(794.1)^2 + (750.6)^2 + (774.7)^2}{9} - \frac{(2319.4)^2}{27} = 105.53.$$

We cannot now isolate the two degrees of freedom for the differences among the center points and the analysis of variance is as follows.

		s.s.	d.f.	m.s.
Due to linear terms		268.36	4	67.09
Due to second order terms		294.92	10	29.492
Residual	126.71	21.18	10	2.118
Blocks		105.53	2	52.765
Total after elimination of mean		689.99		

It is seen that in this example a large proportion of the residual variance is accounted for by the blocks. On the assumption that our model is adequate, the mean square of 2.118 provides an estimate $\hat{\sigma}^2$ of σ^2. This estimate will therefore be employed in calculating the standard errors of the variance coefficients. If extra runs at the center point could be made then an equal number of these should be allocated to each block. The pooled variances for replications at the center point within each block would then provide an estimate of error appropriate for testing the adequacy of the model.

6.4. *Variances, Covariances and Standard Errors.*

The variances and covariances of the various estimates are obtained from the formulae in Table 5a with an appropriate estimate $\hat{\sigma}^2$ of the experimental error variance replacing σ^2 in those formulae. In the present example we employ the estimate $\hat{\sigma}^2 = 2.118$. Taking square roots of the estimated variances we obtain the following values for the standard errors of the estimates:

$$\text{S.E.}(b_0) = \sqrt{\frac{2.118}{3}} = .84;$$

$$\text{S.E.}(b_i) = \sqrt{\frac{2.118}{12}} = .42;$$

$$\text{S.E.}(b_{ii}) = \sqrt{2.118 \cdot \frac{5}{24}} = .66;$$

$$\text{S.E.}(b_{ij}) = \sqrt{\frac{2.118}{4}} = .73.$$

6.5. *General Comments on the Analysis.*

The simple type of analysis illustrated above is appropriate for designs 1, 2, 3, 5, and 8. The analysis of designs 4, 6, 7, 9, and 10 is slightly more complicated. Estimates of b_0, the constant term, and the linear terms b_i are obtained exactly as before. The multiplier D for calculating the interaction effects however takes two values for these designs. The multiplier D_1 is appropriate for these combinations of variables listed as first associates in Table 4 and D_2 for those combinations listed as second associates. In Table 4 combinations belonging to only one of the associate classes are listed. All others belong to the other associate class. For example, in design No. 4 the interactions 1 4; 2 5; and 3 6 are between first associates and take the multiplier D_1. For design No. 7 however it is more economical in space to list the second associates which take the multiplier D_2. In calculating the estimate of b_{ii} (Table 5a), C_1 is the multiplier of $\sum_j^{n_1} \{jjy\}$ in which the j's are first associates of i while C_2 is the multiplier of $\sum_l^{n_2} \{lly\}$ in which the l's are second associates of i.

Appendix A

Derivation of the Class of Three Level Designs

The requirements which need to be satisfied in order that a design shall be second order rotatable are given elsewhere [5]. It is desirable [5] when possible to satisfy the additional condition that biases due to neglected third order terms are zero. The conditions which the design points must then satisfy are as follows:

$$(1) \quad \begin{cases} \sum_{u=1}^{N} x_{iu}^2 = \sum_{u=1}^{N} x_{ju}^2 > 0 & \text{all } i, j \\ \sum_{u=1}^{N} x_{iu}^4 = \sum_{u=1}^{N} x_{ju}^4 > 0 & \text{all } i, j \end{cases}$$

$$(2) \quad \sum_{u=1}^{N} x_{iu} = \sum_{u=1}^{N} x_{iu}^3 = \sum_{u=1}^{N} x_{iu}^5 = 0 \qquad \text{all } i$$

$$(3) \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \sum_{u=1}^{N} x_{ku}^2 x_{lu}^2 > 0 \qquad i \neq j,\ k \neq l$$

$$(4) \quad 3 \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \sum_{u=1}^{N} x_{iu}^4 \qquad i \neq j$$

$$(5) \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju} = \sum_{u=1}^{N} x_{iu}^4 x_{ju} = \sum_{u=1}^{N} x_{iu}^3 x_{ju}^2 = 0 \qquad i \neq j$$

$$(6) \quad \sum_{u=1}^{N} x_{iu} x_{ju} = \sum_{u=1}^{N} x_{iu}^3 x_{ju} = 0 \qquad i \neq j$$

$$(7) \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{ku} = 0 \qquad i \neq j \neq k$$

$$(8) \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju} x_{ku} = 0 \qquad i \neq j \neq k$$

$$(9) \quad \sum_{u=1}^{N} x_{iu} x_{ju} x_{ku} = \sum_{u=1}^{N} x_{iu}^3 x_{ju} x_{ku} = 0 \qquad i \neq j \neq k$$

$$(10) \quad \sum_{u=1}^{N} x_{iu} x_{ju} x_{ku} x_{lu} = 0 \qquad i \neq j \neq k \neq l$$

$$(11) \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju} x_{ku} x_{lu} = 0 \qquad i \neq j \neq k \neq l$$

$$(12) \quad \sum_{u=1}^{N} x_{iu} x_{ju} x_{ku} x_{lu} x_{mu} = 0 \qquad i \neq j \neq k \neq l \neq m$$

Bearing in mind that for our present purpose each x can take only the values -1, 0 or 1, we consider what is implied, first for single columns of the design, then for pairs of columns and so on. In what follows a coincidence means the occurrence of 1's (plus or minus) in the same row of the design matrix. In general where we refer to the occurrence of a "1" we mean a $+1$ or a -1. The equation numbers refer to the appropriate relations above.

(a) Single columns. The same number of 1's occur in each column. Half of these are $+1$ and half -1 (Equations 1 and 2).

(b) Two columns. The number of coincident 1's is greater than zero and the same for all sets of two columns. For these coincident 1's

$$\sum x_{iu} = 0 \quad \text{and} \quad \sum x_{iu}x_{ju} = 0$$

where, here and subsequently, the summation is taken over the relevant coincidences (Equations 3, 5 and 6).

(c) Three columns. For the coincident 1's occurring in any three columns

$$\sum x_{iu} = 0; \qquad \sum x_{iu}x_{ju} = 0; \qquad \sum x_{iu}x_{ju}x_{ku} = 0.$$

(Equations 7, 8 and 9)

(d) Four columns. For the coincident 1's occurring in any four columns

$$\sum x_{iu}x_{ju}x_{ku} \doteq 0; \qquad \sum x_{iu}x_{ju}x_{ku}x_{lu} = 0.$$

(Equations 10 and 11)

(e) Five columns. For the coincident 1's occurring in any five columns

$$\sum x_{iu}x_{ju}x_{ku}x_{lu}x_{mu} = 0.$$

(Equation 12)

Considering the possible designs we see from (b) that we cannot use any arrangement for which no coincidences occur. It is on the other hand possible, in principle, to generate designs in which 1's are coincident only in pairs of columns. In this case requirements (c), (d), and (e) are automatically satisfied. To satisfy requirement (b) consider the coincidence of 1's in the ith and jth column. For these ones we require $\sum x_{iu} = 0$; $\sum x_{ju} = 0$ and $\sum x_{iu}x_{ju} = 0$. The fewest number of coincidences for which this can be satisfied is four. The actual values of the coincident 1's must then be some permutation of the rows of the 2^2 arrangement:

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

We now need to include these component arrangements so that Equation (4) is also satisfied. This requires that the number of coincidences in each pair of columns is one third the number of 1's occurring in each column. The combinatorial properties required of the coincidences are seen to be exactly those of a balanced incomplete block design with $r = 3\mu$ (where, in the incomplete block design, r is the number of times each treatment is replicated and μ is the number of times each pair of treatments appear together in the same block). Precisely this method of construction is employed in design No. 2.

Designs may also be obtained in which 1's are coincident only in sets of three columns. Requirements (d) and (e) are automatically satisfied and requirement (c) can be met by arranging that the actual values of the coincident 1's form the elements of a 2^3 factorial. By arranging once more that the coincidences follow those of a balanced incomplete block design with $r = 3\mu$ all conditions are satisfied. Design No. 5 is an example of this type of arrangement. As has been shown [14], exactly similar arguments may be employed for designs with higher numbers of coincidences. Where coincidences of more than five columns are involved we could satisfy all the requirements with fractional factorials instead of full factorials for the basic units provided that the generators of the fractional factorials contain not less than six elements.

Among the designs listed in Table 4, the above method of generation accounts for arrangement No. 2 for four variables in twenty-four runs and arrangement No. 5 for seven variables in fifty-six runs. Other arrangements of this kind are available, but only those giving low redundancy factors are listed here. Balanced incomplete block designs for which $r = 3\mu$ and for which the redundancy factors are satisfactory are unfortunately not available for all k. To obtain useful designs for other values of k some relaxation in our requirements must be made. A natural modification is to employ balanced incomplete block designs for which $r \neq 3\mu$. It is easily seen that for such designs all the equations (1) through (12), excepting (4), will be satisfied. Instead the design will satisfy

$$\frac{r}{\mu} \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \sum_{u=1}^{N} x_{iu}^4 .$$

The ratio r/μ may be chosen to be as close to 3 as possible. Designs of this class in Table 4 are No. 1 ($k = 3$, $r/\mu = 2$), No. 3 ($k = 5$, $r/\mu = 4$), No. 8 ($k = 11$, $r/\mu = 2.5$). The resulting designs are not quite rotatable but, as has been pointed out already, the property of rotatability is desirable rather than critical and for the designs discussed the variance of $\hat{y}$ at points equidistant from the origin changes little. This is shown quantitatively in the last column of Table 5c which shows the non-sphericity factor "I" for the designs considered [14]. This non-sphericity factor measures the range of variance of $\hat{y}$ divided by its midrange on the unit sphere

$$\sum_{i=1}^{k} x_i^2 = 1.$$

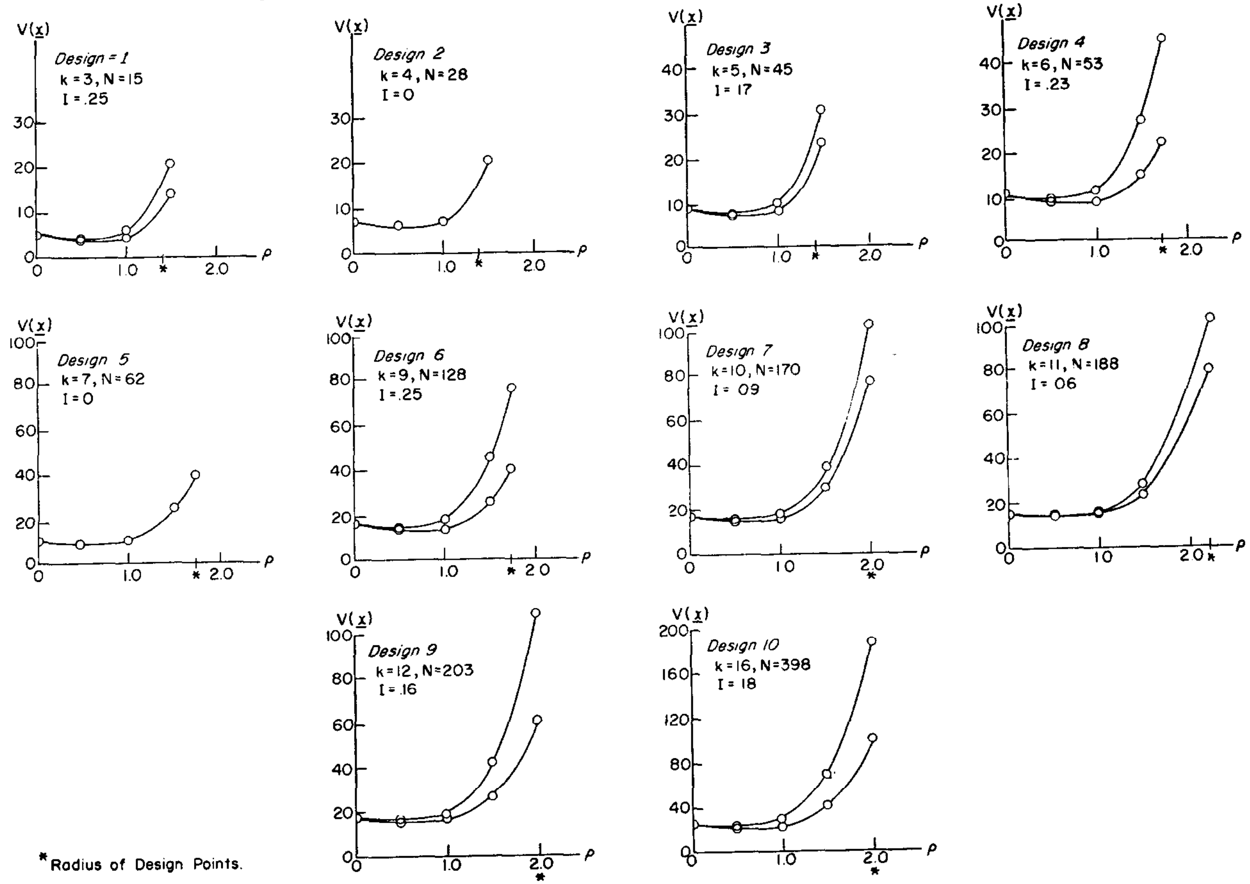
For rotatable designs the factor is zero.

A further relaxation of the same kind is to allow the use of partially balanced incomplete block designs. Again all the conditions will be satisfied except those of equations (3) and (4). Instead of this relationship, we will have for these designs

$$\frac{r}{\mu_1} \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \sum_{u=1}^{N} x_{iu}^4 , \qquad \text{for } i, j \text{ first associates}$$

$$\frac{r}{\mu_2} \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \sum_{u=1}^{N} x_{iu}^4 , \qquad \text{for } i, j \text{ second associates}$$

STANDARDIZED VARIANCE [V(x)] FUNCTIONS



*Radius of Design Points.

where μ_1 is the number of times first associate treatments appear together in the same block and μ_2 is the corresponding parameter for second associates. Once more these designs are nearly rotatable and have low redundancy factors. The values of I and R for these designs also are shown in Table 5c. Characteristic of this classification of designs is that the variances of interaction coefficients (b_{ij}) are different depending upon whether i and j are first or second associates. In practice, as can be determined from the formulae and constants in Table 5a these differences in variance are not serious and the resulting designs are perfectly satisfactory. In Table 4 designs No. 4 (for $k = 6$), No. 6 (for $k = 9$), No. 7 (for $k = 10$), No. 9 (for $k = 12$) and No. 10 (for $k = 16$) are of this type.

Equation (12) of the moment conditions for three-level designs arises from the requirement that biases due to third order terms be made zero. The relaxation of this condition would preserve all the properties of the design except that if, contrary to assumption, three-factor interaction coefficients were not zero, these would cause the two-factor interaction coefficients to be biased. Condition (12) is relaxed in design No. 8 in which a half-replicate of the basis 2^5 design is employed.

Figure 1 gives the variance profiles for the designs of Table 4. These graphs show the standardized variance function

$$V(\mathbf{x}) = \frac{N}{\sigma^2} V(\hat{y})$$

plotted as a function of $\rho = (\sum_i x_i^2)^{\frac{1}{2}}$ the distance from the center of the design. A number of center points have been added to make the variance at $\rho = 0$ equal to the midrange variance at $\rho = 1$ and is close to the number recommended in Table 4. The small adjustment to n_0 required to distribute the center points equally among blocks has a negligible effect on these graphs. For non-rotatable designs the two curves indicate the maximum and minimum variance obtained [14] on a sphere of radius ρ. They thus represent the envelope of all possible variance functions that might be obtained by proceeding from the origin out along any arbitrary radius.

Our object here is merely to present a set of designs whose properties are sufficiently desirable to justify immediate application, it is by no means implied that the designs we have listed are exhaustive. In particular, as will be reported elsewhere, the method of generation here used can provide designs in which the number of ones occurring in each row is not constant. Even within the particular class of designs which we have considered (in which the number of ones in each row is constant), the designs presented are far from exhaustive. A wider but by no means complete selection of such designs is given in [14].

Appendix B

In Section 3.0 the three-level 24-point arrangement is described which forms the basis, with added center points, for a second order rotatable design. As is mentioned in the text, this design is in fact a rotation of the four-variable central composite rotatable arrangement. This may be readily confirmed in the following way.

Upon post-multiplying the matrix (excluding center points) for design No. 2

given in Table 3 by the orthogonal matrix

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

we obtain, except for the scale factor $1/\sqrt{2}$, the design matrix of the rotatable central composite arrangement [5] which may in an obvious shorthand notation be denoted by

$$\begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & \pm 1 \\ \pm 2 & 0 & 0 & 0 \\ 0 & \pm 2 & 0 & 0 \\ 0 & 0 & \pm 2 & 0 \\ 0 & 0 & 0 & \pm 2 \end{bmatrix}.$$

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