Notes

The Sequential Youden Square

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Starting with 2 runs on each of 7 variables, and continuing if necessary through 7 runs, the experimenter can calculate at the end of each run the effects of the 7 variables, and the possibility of learning more with another run.

Any experiment—in plant or laboratory will often embrace more than one shift, or one day, more than one piece of equipment, or more than one animal, more than one batch of raw material, or more than one operator, or more than one plot of ground. Each of these variables may affect the results, thus obfuscating the effects of the design treatments (variables) such as temperature, pressure, or concentration. To avoid this, it is logical to arrange the experiment in BLOCKS, where each Block corresponds to one of the several shifts, or to one of the several batches of raw material, etc. In Statistical Quality Control work, such blocks are Rational Sub-Groups. In experimental design, Blocks are the usual terminology.

The complete experiment will test all of the design treatments in each of the Blocks. The design treatments are arranged in the Blocks in random order, so the whole design is known as a Randomized Block Design. The efficiency of many other designs is determined by comparison with the corresponding Randomized Block. However, there are plenty of times that it is not practical, or not even possible, to test all of the design treatments in each of the Blocks. For instance, if batches of raw material correspond to Blocks, but each batch will support only six plant-scale experiments when you have seven treatments to explore, you can’t use Randomized Blocks. The alternate is Incomplete Blocks.

There are many varieties of Incomplete Blocks, but in chemical experimentation, time trends usually must be expected, so a design that compensates for these as well as for the differences in the Rational Sub-Groups has an advantage over the others. Such a design is a type of Balanced Incomplete Block known as a Youden Square.

There are lots of Youden Squares to choose from too, some with 3 Treatments per Block, some with 4, 5, etc. Considering the simplest case—3 Treatments per Block, 4 design variables (Treatments) make a nice design requiring

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just 12 runs. However, if 5 Treatments are to be run in Blocks of size 3, 10 Blocks are required in order that each pair of Treatments occurs in the same number of Blocks, and that all Treatments occur the same number of times in each Column. This requires 30 runs. A 6 × 3 takes 90 runs. On the other hand, a 7 × 3 is balanced with only 21 runs. Thus, if you have more than 4 Treatments, use the 7 × 3 even if you have to pad the design by running one or two Treatments twice, or by including a dummy variable or two. At 4 Treatments per Block, the 5 × 4 requires only 20 runs, but the 6 × 4 needs 120 while the 7 × 4 is balanced as just 28. There's something about a 7.

Not all statistical tests show both the Youden 7 × 3 and 7 × 4, but those that do present them essentially as given in Table I. Every Column contains every Treatment, giving the time-trend compensation if Columns correspond to Time. Every possible pair of Treatments occurs within one Block or another—one each for the 7 × 3 and twice each for the 7 × 4.

This introduces another consideration: the power of the design. For 28 runs instead of 21—an increase of 33%—each pair is found in a Block twice instead of once—an increase of 100%. The Efficiency of the 7 × 3 (as compared with a Randomized Block of 7 Treatments) is 78%, while the 7 × 4 rates 89%.

Wouldn't it be nice if we could run a 7 × 3, calculate the results quickly, and determine whether or not a fourth run would clarify the situation, or would merely make what is already definite a bit more so? The way the designs are shown here, however, you can make the first two experiments (Columns) without committing yourself, but if the third starts with D, you are blocked from extending to the 7 × 4. On the other hand, if the third Column starts with a C, you must finish all 4 Columns in order to calculate the results with any precision.

A chance remark by H. O. Hartley (of Texas A. & M.) struck a spark in the minds of Otto Dykstra (American Cyanamid) and the author. A very simple change will solve our problem, as shown in Table II. Now the first three Columns constitute a perfect 7 × 3, and the fourth Column converts the design to a perfect 7 × 4. (Incidentally, Cochran and Cox [1] give the 7 × 4 as shown here, but the 7 × 3 as shown in Table I.) Thus the 7 × 3 is imbedded in the 7 × 4.

### Table I

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<thead>
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<th>Column Block</th>
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<td>7</td>
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</tr>
</tbody>
</table>
But why settle for a 7 X 3 and a 7 X 4? Let's start with a 7 X 2—the first two Columns. Without changing anything but the appearance, these Columns may be written as in Table III. This design is known as a "Round Robin" because it begins where it ends—with Treatment A, in this case. It is not a Balanced Incomplete Block for all possible pairs do not occur an equal number of times. However, it is a handy place to start the Sequential Youden Square. The method of analysis is given [2]. This design gives no estimate of error. However, we usually have an estimate of that from previous work, so we can test the Treatment Effects and Block Effects for significance at this early stage of the design.

The full Sequential Youden is shown in Table IV. We start with the first (left-hand) Column for the first experiment. If some Treatment is unworkable, or some Block utterly different from the rest, we can sense this now, and modify our plans. If the first Round appears to be successful, we go to the second, calculating our results as a Round Robin. Assuming that the quantity we are using for Error is reasonably correct, we can see whether any of our Treatments is significantly different from the others. If not, we can calculate how many more runs would be needed to make the difference we see now truly significant.
(assuming our present estimate of Treatments is correct). If more than 7 Rounds are needed, we can stop now. Otherwise, we continue.

By the finish of the third Column we have a Youden $7 \times 3$, which can be analyzed by standard methods. (See, for instance [2].) Now we have also an independent estimate of Error, which we can use as is, or combine with our previous, external estimate. Again we can see what Treatments are significantly different, what ones are not and never will be, and what ones are not, but may be if we continue.

Assume we decide to continue. The fourth Column completes a Youden $7 \times 4$, which is analyzed by the same procedure as was used for the $7 \times 3$. Our estimate of Error is improving, for we have more degrees of freedom for Error (12 now, versus 6 for the $7 \times 3$). Thus our estimate of “significant” is sharpening.

If there is a good probability that we shall learn more by continuing we run Column V. Now we have no single, neat design, but the three Columns headed by C-E-F, by C-D-F, and E-F-A are all Youden $7 \times 3$'s, which can be analyzed as before.

Again, if there is reason to, we can continue. Column VI fills out a Youden $7 \times 6$—analyzable by the same methods as were used for the $7 \times 3$ and the $7 \times 4$. And, finally, Column VII completes a Latin Square, whose analysis is given in [1], Par. 4.34. Thus, Round by Round, we can feel our way, stopping when we have learned all we need, or see no hope of learning more by further effort.

Every time this design has been used for serious business, we have learned all we have hope of learning by the fourth Column, if not by the second or third. Thus there is no practical example of an experiment that had to be carried the whole way through the Latin Square stage. To test the general concept, however, a colleague—Pat Opsahl—and I devised a trivial study that would not waste our employer's money if continued to the end. We made up 7 nail-polish formulas: Brands A and B, with and without Base Coat or Top Coat, and one Brand with both Base and Top Coats. The "Blocks" were the five fingers of Pat's right hand, and the thumb and index of her left hand. Every
Monday morning, Pat would clean her nails thoroughly, then apply the appropriate Treatment to the appropriate Block (finger). She then observed the number of days elapsing before each Treatment chipped. The longest life observed was 7 days, so she was always ready for the next Round on the following Monday morning.

The first Round gave results varying between 1 day and 7 days, so we had real hope of establishing some significant differences by at least the end of Round VII. Maybe the Treatment that failed in 1 day was actually unworkable, but we took a chance, and continued.

At the end of Round II we did not have a real estimate of Error, of course, but guessing that certain Treatments would give similar results, we approximated the Standard Deviation as 2 days. By the end of the 7th week we had estimated the Standard Deviation, with 30 degrees of freedom, as 1.67, so our original estimate was not bad. At this stage, it looked as though Formula A was about as good as Formula B (though one costs considerably more than the other). Base Coat seems to help, but not Top Coat. Fingers (Blocks) are apparently important, the contrast between the two fingers of the left hand with the corresponding fingers of the right hand being most promising. These conclusions were made more quantitative by the subsequent Rounds, but were not changed qualitatively, except for the effect of Base Coat.

By the end of the third week, there were still no significant differences, but we calculated that Round IV would really establish the difference between the left and right hands. We also predicted that no reasonable amount of testing would show significant differences between Formulas, or between the effects of Base and/or Top Coats versus straight polish. These predictions were confirmed exactly, even after the seventh Round. Thus this experiment, like all of our other applications of the Sequential Youden Square, really terminated with the fourth Round. However, this time we pushed through to the bitter end, and confirmed what had been predicted.

As a matter of interest, we mention Columns. At the outset of this paper, we related Column Effects to time trends, but in this “experiment” Columns became discontinuous shifts in base line. The mathematics of the Youden Square, however, permitted correct evaluation of Treatments and Blocks in spite of this jumping around in Columns. The first 4 weeks were uneventful; the Column Totals were 22, 25, 26, and 22. During the fifth week, however, Pat entertained, and this entailed dishwashing. The Column Total dropped to 13. Obviously, being a Sohio Chemist is better than being a housewife as far as life of finger-nail polish is concerned. And the sixth week Pat spent on Cape Cod, taking her seven little vials with her. Sun, sea, and sand took its toll; the Column Total was 9. For the last week, Pat was a chemist again, and the Column Total was 23.

To summarize the results at the end of Round VII, the analysis of variance of the Latin Square gave the following “F” tests:

for Polishes 0.95 a toss-up.
for Columns 2.08 between 5% and 10% significant.
for Fingers 3.25 between 1% and 2½% significant.
Columns approached significance only after the fifth Round, but for the other two variables, here are the “$F$” values at the end of Round IV:

- for Polishes $1.03$ a toss-up.
- for Fingers $3.75$ significant at the 2-1/2\% level.

Again, 4 Rounds were enough, and we knew it at the time.

We stress the basic principle rather than the amusing illustration, however. By the use of the Sequential Youden Square, we can feel our way for up to 7 experiments, stopping whenever the predictions at that point say that we have learned all that we are going to. A non-sequential design has to be run as is, and then you're done, no matter what might seem to be just around the corner.

**References**


The Inverse of a Finite Toeplitz Matrix

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SUMMARY AND INTRODUCTION

The inverse of this type of matrix is required in various areas of application of statistical, stochastic control and communication theory. Examples range over i) the fitting of autoregressive series (Siddiqui, 1958; Grenander and Rosenblatt, 1957), ii) the fitting of regression functions when the errors are serially correlated (Hannan, 1960; Whittle, 1963), iii) the estimation of shaping and matching filters in communication theory (Robinson, 1966; Wiener, 1949), iv) adaptive estimation of control system parameters by Bayesian methods (Aoki, 1967).

Whenever any of the above applications require large order matrix inversion of a Toeplitz matrix and require it dynamically for the purpose of updating parameters, then it is important to reduce the amount of computation wherever feasible.

It is the purpose of this note to give new matrix forms for the inverse of a finite Toeplitz matrix which produce a definite saving in computation.

CASE I—MATRIX \( R \) OF EVEN ORDER \((n = 2m)\)

We write the matrix \( R = [r_{i-j}] \) \((i, j = 1, \cdots, n; r_0 = 1)\) in the partitioned form

\[
R = \begin{bmatrix}
A & BA_v \\
A_vB & A
\end{bmatrix}
\]

where

\[
A = \begin{bmatrix}
1 & r_1 & \cdots & r_{m-1} \\
r_1 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
r_{m-1} & \cdots & \cdots & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
r_{2m-1} & \cdots & r_m \\
\vdots & \ddots & \vdots \\
r_m & \cdots & r_1 \\
1 & \cdots & 0
\end{bmatrix}, \quad A_v = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 1
\end{bmatrix}
\]

If we now write the matrix equation for the inverse of \( R \) as

\[
\begin{bmatrix}
A & BA_v \\
A_vB & A
\end{bmatrix}\begin{bmatrix}
C & D \\
D' & E
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

where \( I \) is the unit matrix, then by straightforward matrix algebra or using

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Schur's Identity (4) we can determine C, D, and E. The result is given by

\[ R^{-1} = \begin{bmatrix} \Delta^{-1} & -A^{-1}BA^{-1}A_s \\ -A_s\Delta^{-1}BA^{-1} & A_s\Delta^{-1}A_s \end{bmatrix} \]

where

\[ \Delta = (A - BA^{-1}B). \]

It should be noted that by expressing the inverse in this way we have effectively reduced the problem from one of inverting a 2m X 2m symmetric matrix to the inversion of two m X m order matrices as well as three matrix multiplications.

Now we can take further advantage of the special structure of this matrix and through the use of the permutation matrix \( A_s \) express the inverse \( R^{-1} \) in the following form

\[ R^{-1} = \begin{bmatrix} C & D \\ D' & E \end{bmatrix} \]

where

\[ C = ((A + B)^{-1} + (A - B)^{-1})/2, \]
\[ D = ((A + B)^{-1} - (A - B)^{-1})A_s/2, \quad \text{and} \quad E = A_sCA_s. \]

We have now reduced the scale of computation to merely the inversion of two m X m matrices without any matrix multiplication. The pre-multiplication of a matrix by \( A_s \) reverses the rows of that matrix first to last and vice versa, second to second last and vice versa, etc., whilst post-multiplication by \( A_s \) reverses the columns of the matrix. Thus these actions are merely a rearrangement of the elements of a matrix and not matrix multiplication in the computational sense.

By way of comparison and also justification for our particular form of partitioning of the matrix \( R \) utilising the permutation matrix \( A_s \), we shall consider the inverse of the following partitioned form of \( R \)

\[ R = \begin{bmatrix} A & F \\ F' & A \end{bmatrix} \]

which merely recognises its symmetric nature and equal diagonal sub-matrices but not its banded structure. The inverse of this form leads to

\[ R^{-1} = \begin{bmatrix} A^{-1}F\nabla^{-1}AF^{-1} & \nabla\nabla^{-1} \\ -\nabla^{-1}F'A^{-1} & \nabla^{-1} \end{bmatrix} \]

where

\[ \nabla = A - F'A^{-1}F. \]

We note here that this involves the inversion of two m X m matrices but five matrix multiplications.
CASE II—MATRIX $R$ OF ODD ORDER ($n = 2m + 1$)

In this case we partition the matrix $R$ into the form

$$
\begin{bmatrix}
A & A \alpha x & BA \alpha \\
x' A \alpha & 1 & x' \\
A \alpha B & x & A
\end{bmatrix}
$$

where

$$
A = \begin{bmatrix}
1 & r_1 & \cdots & r_{n-1} & \\
r_1 & 1 & & \\
& \ddots & \ddots & \ddots & \\
r_{n-1} & \cdots & 1 & 
\end{bmatrix}, \\
B = \begin{bmatrix}
r_{2m} & \cdots & r_{m+1} & \\
\vdots & \ddots & \ddots & \\
\vdots & & \ddots & \\
r_{m+1} & \cdots & r_{m} & 
\end{bmatrix}, \\
z = \begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_m 
\end{bmatrix}, \\
A_\beta = \begin{bmatrix}
0 & \cdots & 1 & \\
\vdots & \ddots & \ddots & \\
\vdots & & \ddots & \\
0 & 1 & \cdots & 
\end{bmatrix}
$$

The matrix equation for the inverse takes the form

$$
\begin{bmatrix}
A & A \alpha x & BA \alpha \\
x' A \alpha & 1 & x' \\
A_\alpha B & x & A
\end{bmatrix}
\begin{bmatrix}
C & A_\alpha y & D \\
y' A \alpha & u & y' \\
D' & y & E
\end{bmatrix}
= \begin{bmatrix}
I & (0) & 0 \\
(0)' & 1 & (0)'
\end{bmatrix}
$$

where $(0), (0)'$ denote an $(m \times 1)$ column vector of zeros and an $(1 \times m)$ row vector of zeros respectively.

Again by the same sort of matrix algebra as in the even case we obtain the following form of the inverse $R$ as

$$
R^{-1} = \begin{bmatrix}
C & A_\alpha y & D \\
y' A \alpha & u & y' \\
D' & y & E
\end{bmatrix}
$$

where

$$
M = (A + B - 2A \alpha x' A \alpha), \\
C = (M^{-1} + (A - B)^{-1})/2, \\
E = A_\alpha C A \alpha, \\
D = (M^{-1} - (A - B)^{-1})A_\alpha /2, \\
u = 1 + 2x' A_\alpha M^{-1} A \alpha, \\
y = -A_\alpha M^{-1} A \alpha.
$$

COMMENTS

It is believed that these matrix expressions for the inverse of a finite Toeplitz matrix are new and will provide a very useful saving in computation wherever they are used in practical application. They may also be of assistance to the theoreticians in the analysis of statistical distributional problems.
ACKNOWLEDGMENTS

The author is very grateful to the referee for suggestions which led to a revision of and a reasonable justification for the matrix expressions for the Toeplitz inverse given in this paper.

REFERENCES

On the Choice of Regression in Linear Calibration. Comments on a paper by R. G. Krutchkoff

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In a paper by R. G. Krutchkoff, "Classical and Inverse Regression Methods of Calibration," two methods of calibration are compared by means of the Monte Carlo technique. He comes to the conclusion that the inverse approach to the calibration problem has a uniformly smaller mean square error than the classical approach. Our theoretical treatment of the problem shows that the conclusion is true only for small samples. It is shown that when more than one observation is made on y the advantage of the inverse approach is reduced. A large sample formula for the ratio of the two mean square errors is derived which mainly conforms to the result by Krutchkoff.

1. Introduction

Krutchkoff uses the following model:

\[ y = \alpha + \beta x + \epsilon, \]

where \( y \) is measured with the instrument which we want to calibrate, \( x \) is the variable which we in reality want to measure and which is given fixed and known values by the investigator at the instant of calibration, and \( \epsilon \) is a measurement error. The measurement errors are supposed to be independent and normally distributed with conditional expectation \( E(\epsilon \mid x) = 0 \) and conditional variance \( D^2(\epsilon \mid x) = \sigma^2 \).

The common method of calibration is to estimate \( \alpha \) and \( \beta \) by least-squares fitting of the equation \( y = \alpha + \beta x \) to the data of calibration, where \( a \) and \( b \) are estimates of \( \alpha \) and \( \beta \) respectively. The unknown value of \( x \) is then calculated from an observed \( y \), denoted \( Y \), by the equation:

\[ X = \frac{Y - a}{b}. \]

This is the "classical approach" according to Krutchkoff.

By the "inverse approach" he means that the equation \( x = c + dy \) is fitted to the data of calibration by the method of least-squares. And as an estimate of \( x \) one uses:

\[ X_I = c + dy. \]

Krutchkoff compares these two methods using rather small samples. The largest sample size is 20 observations. The estimator (1.3) has in most cases the smallest mean square error.

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2. A COMPARISON OF THE TWO METHODS \textit{When} \( \alpha \text{ and } \beta \text{ are known} \)

Krutckoff considers only small samples. However, the principal difference between the two methods is best illustrated when the constants \( \alpha \text{ and } \beta \) are supposed to be known.

The "classical estimator" is

\[
X_c = \frac{Y - \alpha}{\beta} \tag{2.1}
\]

which is unbiased, having the conditional variance:

\[
D^2(X_c \mid x) = \left( \frac{\sigma}{\beta} \right)^2. \tag{2.2}
\]

In order to apply the estimator (1.3) we assume that the values of \( c \) and \( d \) have been obtained in a very large experiment with a specific set of \( x \) values with mean and mean squared values denoted by \( \bar{x} \) and \( s_x^2 \), respectively.

Then the coefficients in equation (1.3) can be written as follows:

\[
c - \bar{x} - d(\alpha + \beta \bar{x}) \tag{2.3}
\]

and

\[
d = \frac{1}{\beta(1 + \phi)}, \tag{2.4}
\]

where

\[
\phi = \left( \frac{\sigma}{\beta} \right)^2 \frac{1}{s_x^2}. \tag{2.5}
\]

Substituting for \( c \) and \( d \) in (1.3) gives

\[
X_f = \bar{x} + \frac{Y - \alpha - \beta \bar{x}}{\beta(1 + \phi)} . \tag{2.6}
\]

The estimator (2.6) is biased and we denote the bias by \( B \).

\[
B = E(X_f - x \mid x) = -\frac{\phi}{1 + \phi} (x - \bar{x}) . \tag{2.7}
\]

The conditional variance of \( X_f \) at a given \( x \) is

\[
D^2(X_f \mid x) = \left( \frac{\sigma}{\beta} \right)^2 \frac{1}{(1 + \phi)^2} . \tag{2.8}
\]

From (2.7) and (2.8) we get the mean square error of the estimator \( X_f \):

\[
E\left\{ (X_f - x) \cdot | x \mid \right\} = \left( \frac{\sigma}{\beta} \right)^2 \frac{1}{(1 + \phi)^2} \left[ 1 + \phi \frac{(x - \bar{x})^2}{s_x^2} \right]. \tag{2.9}
\]

We denote the quotient of the mean square errors by \( R(x) \):

\[
R(x) = \frac{E\left\{ (X_c - x) \cdot | x \mid \right\}}{E\left\{ (X_f - x) \cdot | x \mid \right\}} = \frac{(1 + \phi)^2}{1 + \phi \frac{(x - \bar{x})^2}{s_x^2}}. \tag{2.10}
\]
$R(x)$ is symmetric around its maximum, at $x = \bar{x}$ and approaches zero when $|x - \bar{x}|$ increases. $R(x)$ is equal to 1 for

$$x = \bar{x} \pm s_x \sqrt{2 + \phi}$$  \hspace{1cm} (2.11)

When the deviation of $x$ from $\bar{x}$ is smaller than $s \sqrt{2 + \phi}$, the estimator $X_f$ has a smaller mean square error than $X_e$. For larger deviations the relation is reversed.

If more than one observation of $y$ is made at a fixed but unknown $x$ we get, instead of (2.10):

$$R(x) = \frac{(1 + \sqrt{2 + \phi})^2}{1 + m \phi \frac{(x - \bar{x})^2}{s_x^2}}$$  \hspace{1cm} (2.12)

where $m$ is the number of observed $y$'s.

The interval in which $R(x) > 1$ is reduced to:

$$|x - \bar{x}| < \frac{s_x}{\sqrt{m}} \sqrt{2 + \phi}.$$  \hspace{1cm} (2.13)

3. A Comparison of the Two Methods When $\alpha$ and $\beta$ are Unknown

We assume that the calibration is based on a sample of size $n$. If the relative error of $b$, i.e. $\sigma_b / \beta$, and the value of $\phi$ are small it can be shown, when neglecting terms of higher degree than $n^{-1}$, that the quotient between the mean square error of $X_e$ and $X_f$ is

$$\frac{R_e(x)}{R_f(x)} = \left\{ 1 + \frac{1}{n} \left[ 1 + \frac{(x - \bar{x})^2}{s_x^2} \right] (1 + g)^3 \right\} \left/ \left\{ 1 + g^2 k^{-2} (x - \bar{x})^2 \right\} \right.$$  

$$+ \frac{1}{n - 1} \left[ 1 + g (1 + g^2) + 4 g (1 + g)^{-2} \right.$$  

$$+ g k^{-1} (1 + g^2) - 4 g^2 k^{-2} (1 + g)^{-2} (x - \bar{x})^2 \right\}^{-1},$$  \hspace{1cm} (3.1)

where

$$g = \frac{n - 1}{n} \left( \frac{\sigma_b}{\beta} \right)^2 \frac{1}{s_x^2} \quad \text{and} \quad k = \left( \frac{\sigma_b}{\beta} \right)^2.$$

In order to make a comparison with the results by Krutchkoff we have computed $R_e(x)$ for three different values of $n$, namely $n = 6, 10$ and 20. The values of the other parameters are the same as those used by Krutchkoff in his Table V. The computed values as well as the corresponding values obtained by Krutchkoff are given in our Table 1.

The difference between simulated and computed values is small except for large $x$'s. However, for $x = 2.0$ and $n = 20$ the difference is also small. One explanation of this inconsistency is that (3.1) is a large-sample formula. It is remarkable, however, that in almost all tables which Krutchkoff presents, the mean square errors of the two methods are practically unchanged from $x = 2.0$ to $x = 10.0$. 
From Table 1 it is seen that $R_n(x)$ is larger than 1 even at large $x$-values, when $n$ is small. But for $n = 20$ the quotient is smaller than 1 at $x = 2.0$.

Whether $R_n(x)$ is larger than 1 depends also on the value of $k = \sigma/\beta^2$. This is seen from Fig. 1, which shows how the largest root of the equation $R_n(x) = 1$ depends on $n$ and $k$; there are two roots which are symmetric around $x = \bar{x} = 0.5$.

For each $k$ the root of the equation is given as a function of the sample size $n$. The equation has no real root unless the sample size is large enough. When $k = 0.10$ for instance the sample size has to be larger than 30. The figure 30 should not be taken too seriously. We have to remember that Fig. 1 is based on the large sample formula (3.1).

### Table 1

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$k =$ result by Krutchkoff, Table V.

$\circ =$ computed according to formula (3.1).

1) The formula (3.1) is probably too rough an approximation for large $x$-values, at least for small $n$. 

Figure 1—The root of $R_n(x) = 1$ as a function of the sample size for different values of $k$. The equation has no real root unless the sample size is large enough.
That no solution to the equation $R_\alpha(x) = 1$ exists means that in small samples the "inverse approach" to the calibration problem has a uniformly smaller mean square error than the "classical approach". This is also the conclusion reached by Krutchkoff, but he does not say that it is only true for small samples.

**Reference**

Combining Independent Estimators—
An Empirical Sampling Study

PAUL S. LEVY
Boston City Hospital and Harvard Medical School

1. Introduction

Suppose there exist $k$ independent unbiased estimators, $X_i$, of a parameter, $\mu$, where each $X_i$ is the arithmetic mean of $n_i + 1$ observations normally distributed with mean $\mu$ and unknown variance, $\sigma_i^2$. Furthermore, suppose that the usual mean squares, $s_i^2$, $i = 1, \ldots, k$, are available as estimators of the unknown variances, $\sigma_i^2$.

The problem of combining the $X_i$ into a single estimator of $\mu$ has been treated at length in the literature, especially by Cochran (1937 and 1954), and the latter gives a detailed review of the literature on this problem. If one wishes to avoid the more complex methods of combining the $X_i$ into a single estimator, i.e., methods which involve partial weighting (Yates and Cochran, 1938), preliminary testing (Cochran, 1954), or Bayesian methods, and if one wishes to take into consideration the possible inequality of the $\sigma_i^2$, one is left with two methods of combining the $X_i$, namely 1) methods based on the likelihood equation or a modification of it, of which the maximum likelihood estimator, $\hat{\mu}$, is the most popular, and 2) the weighted mean $\bar{\mu}$:

$$\bar{\mu} = \frac{\sum_{i=1}^{k} \left( \frac{n_i + 1}{s_i^2} \right) X_i}{\sum_{i=1}^{k} \frac{n_i + 1}{s_i^2}}.$$

Of the two, $\bar{\mu}$ is used in practice much more frequently than $\hat{\mu}$, possibly because 1) $\hat{\mu}$ must be obtained numerically by iterative methods and 2) there exist approximations for the variance of $\bar{\mu}$ derived by Meier (1953) and modified by Cochran and Carroll (1953) which are valid for all $k$ while formulae for variance of $\hat{\mu}$ are asymptotic in $k$. With the wide availability of electronic computers, $\bar{\mu}$ can now be computed in a matter of microseconds once an algorithm is programmed, so that computational convenience alone should not influence the use of $\bar{\mu}$ over $\hat{\mu}$. It is the purpose of this note to show that over a wide variety of situations, $\bar{\mu}$ has higher precision than $\hat{\mu}$, and also to show that formulae for the variance of $\bar{\mu}$ which were derived under the assumption of large $k$ can be used with impunity for small $k$.

Received June 1968; revised Oct. 1968.

* Neyman and Scott (1948) discuss a general class of consistent estimators of $\mu$ which can be derived by modification of the likelihood equation, and have produced in this way an estimator of $\mu$ which has smaller asymptotic variance than the maximum likelihood estimator, $\hat{\mu}$.
2. Methods

A sampling study was performed on the IBM 7094 computer at the Harvard Computation Center. The algorithm programmed in Fortran IV can be summarized as follows: For each sample realization, a pseudo random normal generator was used to generate random normal variables with mean 0 and variance 1. From these, appropriate $X_i$ and $s_i^2$ were constructed along with the weighted mean, $\bar{X}_w$. Finally the maximum likelihood estimate, $\hat{\mu}$, was obtained iteratively by the Newton-Raphson method from the equation

$$
\sum_{i=1}^{k} \frac{(n_i + 1)(X_i - \hat{\mu})}{n_i s_i^2 + (n_i + 1)(X_i - \hat{\mu})^2} = 0
$$

which is derived from the logarithm of the likelihood of the sample in the usual way.

Each sample realization, therefore, yields a value of $\hat{\mu}$ and $\bar{X}_w$. For each set of constants $\{k, \sigma_i^2, n_i, i = 1, \cdots, k\}$ 3000 sample realizations were obtained as above, and from these, the empirical variance of $\bar{X}_w$ and the empirical mean square error of $\hat{\mu}$ were computed. The computer time needed to perform these calculations was approximately proportional to $k$ and to the average of the $n_i$'s. It varied from less than a minute for $k = 3$ and all $n_i = 1$ to 5.5 minutes for $k = 12$ and all $n_i = 10$.

The sampling experiments were designed primarily to investigate the influence on $R$, the ratio of the empirical variance of $\bar{X}_w$ to the empirical mean square error of $\hat{\mu}$, of inequality of the $n_i$'s, average $n_i$, and number of estimators, $k$, being combined. To accomplish this, a grid was constructed on which each point was defined by the couplet $(n_m, \bar{n})$ where $n_m$ is the smallest $n_i$ and $\bar{n}$ the average $n_i$ among the $k$. In the simulation, one-third of the $s_i^2$ were given $n_m$ degrees of freedom, one-third given $\bar{n}$ degrees of freedom, and the remaining third given $2\bar{n} - n_m$ degrees of freedom. For example, suppose $k = 6$ and the point on the grid being considered is defined by $n_m = 5$, and $\bar{n} = 7$. This means that 2 of the 6 $s_i^2$ have 5 degrees of freedom, 2 have 7 degrees of freedom and 2 have 9 degrees of freedom. All the $\sigma_i^4$ were set equal to 1 so that the effect on $R$ of inequality of the $n_i$ would not be confounded with that of inequality of the $s_i^2$.

An expression for the variance of $\bar{X}_w$ valid for $k = \infty$ was derived by Cochran and Carroll (1953) and a formula for the variance of $\hat{\mu}$ was derived by Neyman and Scott (1948). These are

$$
V(\bar{X}_w) = \frac{\sum_{i=1}^{k} n_i^2 w_i}{(n_i - 2)(n_i - 4)} \left\{ \sum_{i=1}^{k} \frac{n_i w_i}{n_i - 2} \right\}^2
$$

(valid for $n_i \geq 5, i = 1, \cdots, k$) and

$$
V(\hat{\mu}) = \frac{\sum_{i=1}^{k} n_i^2 + 1 w_i}{\left( \sum_{i=1}^{k} n_i \right)^2}
$$
where
\[ w_i = \frac{1}{\sigma_i^2}, \quad i = 1, \ldots, k. \]
Thus, for the portion of the grid in which \( n_u \geq 5 \), we can obtain the ratio of the asymptotic \( (k = \infty) \) variance of \( \bar{X}_w \) to that of \( \bar{X} \) by dividing (2) by (3).

3. RESULTS

Table 1 shows \( R \) as a function of \( n_u \) and \( n \) for \( k = 3, 6, 12 \) and the ratio of formula (2) to formula (3) for \( k = \infty \). The points in the table greater than one indicate that \( \bar{X}_w \) has greater precision than \( X \). Entries in the main diagonal are points at which all the \( n_i \) are equal. The trends noted here are that 1) if \( k \), the number of estimators being combined, is large, there may be considerable gain in precision by use of \( \bar{X}_w \) to \( \bar{X} \), 2) for a given \( n \), the relative precision of \( \bar{X}_w \) to \( \bar{X} \) improves as the \( n_i \) become more diverse (i.e., the smallest entries in the vertical columns of Table 1 are those along the main diagonals).

If \( \bar{X} \) is to be of practical use, some valid estimate of its variance should be available. Table 2 gives the ratio of the asymptotic variance of \( \bar{X} \) (formula (3)) to the empirical mean square error of \( \bar{X} \) over the portion of the grid for which \( n_u > 1 \). Over the range of \( n_i \) sampled, the agreement of formula (3) with the empirical mean square error is quite good for \( k = 12 \) and \( k = 6 \), and except for the case in which all \( n_i = 3 \), is reasonably good (and always conservative) for \( k = 3 \).

4. SUMMARY

An empirical sampling study compares the weighted mean and the maximum likelihood estimator with respect to precision in the problem of combining unbiased estimators of a parameter \( \mu \) having unknown variances. Over the sample space investigated, the maximum likelihood estimator generally showed higher precision than the weighted mean. The formula derived by Neyman and Scott (1948) for the asymptotic variance of the maximum likelihood was a reasonably good approximation of its empirical variance in most cases, even if \( k \), the number of estimators being combined, was small.

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Table 2

The Ratio of the Asymptotic Variance (for \( k = \infty \)) of \( \hat{\beta} \) to the Empirical Mean Square Error of \( \hat{\beta} \) for Selected Values of \( n \) and \( n_{\text{eff}} \), for \( k = 3, 6, \) and 12; and for all \( \sigma^2_i \) equal.

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Acknowledgments

This study was supported in part by General Research Support, Boston City Hospital, and in part by research grant no. IIID-02188, National Institute Child Health and Human Development, U.S.P.H.S. In addition, the author wishes to thank John Gart for introducing him to the problem of combining estimators and for much helpful advice on this topic over the last 5 years.

References


Comparisons of Approximations to the Percentage Points of the Sample Coefficient of Variation

BORIS IGLEWICZ* AND RAYMOND H. MYERS†

Case Western Reserve University and Virginia Polytechnic Institute

I. INTRODUCTION

It may be of some interest in certain statistical problems to have available percentage points of the sample coefficient of variation (S.C.V.). If the parent population is assumed to be normal, the obtaining of such percentage points from tables of the non-central t distribution might be thought to be simple. However, one does encounter difficulty in using these tables to calculate the percentage points of the S.C.V. One can, as an alternative, use the exact table of Iglewicz (1967).

A number of approximations for the percentage points of the S.C.V. exists in the literature. These approximations may be useful as alternatives to either of the tables mentioned above. The applicability of several of these approximations is further enhanced by their simplicity. In this paper we compare these approximate results with the exact ones in order to determine the quality of the approximations.

II. NOTATION

Throughout this paper it is assumed that one has a random sample \( \{X_i\}_{i=1}^n \) of normally and independently distributed random variables, i.e., \( X_i \sim N(\mu, \sigma^2) \). Furthermore let

\[
S = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n - 1}} ;
\]

\( v = \frac{S}{\bar{X}} \), the S.C.V.; \( V = \frac{\sigma}{\mu} \), the C.V. and \( v_n = \left( \frac{n - 1}{n} \right)^{1/2} \).

III. APPROXIMATIONS

The approximations to be discussed are given below:

1. McKay’s Approximation (M).

McKay (1932) gave the following approximate distribution of \( v_n^2 \):

\[
B\left[ \frac{v_n^2}{1 + v_n} \right] \sim \chi^2_{n-1}
\]  

(3.1)

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† Now at Virginia Commonwealth University, Department of Biometry.

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where

\[ B = n \left( 1 - \frac{1}{\sqrt{\pi}} \right) \]

It must be assumed at this point that the probability of a negative \( v \) (corresponding, of course, to the probability of a negative \( \bar{X} \)) is quite small. On the basis of this assumption, it follows from (3.1) that approximately

\[ \Pr \left( B \left( \frac{v^2}{1 + v^2} \right) < \xi \mid V \right) = \Pr \left( v < \left( \frac{n}{n - 1} \left( \frac{\xi}{R - \xi} \right) \right) \mid V \right) \]

(3.2)

Note that \( v^2/1 + v^2 < 1 \) implies that \( B > \xi \). One can consequently use equation (3.2) to obtain the approximate percentage points of \( v \).

(2) Hald’s Approximation (\( H \)).

By again making the assumption that the probability of a negative \( v \) is negligible, we can write

\[ \Pr (v \leq t_0 \mid V) \approx \Pr (Z \leq 0 \mid V) \]

where \( Z = S - t_0 \bar{X} \). The statistic \( Z \) then can be used for obtaining approximate percentage points of \( v \). Approximations to the mean and variance of \( Z \) are

\[ \mu(Z) = \sigma - \mu_{t_0}, \quad \sigma^2 = \frac{1}{2(n - 1)} + \frac{t_0^2}{n} \]

(3.3)

respectively. Suppose we assume, in order to obtain a useful approximation, that \( Z \) is a normal variate, and call \( t_0 = t_0 \) the \( p \)th percentile point of \( v \) and \( u_0 \), the \( p \)th standard normal percentile point, then

\[ t_0 \approx V \left\{ 1 + \frac{u_0}{\left( \frac{1}{2(n - 1)} + \frac{t_0^2}{n} \right)^{1/2}} \right\} \]

(3.4)

where \( C = 1 - V^2 u_0^2 / n \). This approximation can be found in Hald (1952).

Further approximations are based on asymptotic expansions for the mean, second, third, and fourth central moments of \( v \). [See Iglewicz (1967)].

(3) Pearson’s moment fit approximation (\( P \)).

The approximate moments were used to fit Pearson distributions to \( v \) and approximate percentile points of the S.C.V. were obtained from the table of Johnson, et al (1963).

(4) S.C.V. assumed normal (\( N_1 \)).

The S.C.V. was assumed normal with the mean and variance given by expansions to order \( 1/n^3 \).

(5) S.C.V. assumed normal (\( N_2 \)).

The S.C.V. was again assumed normal, but the approximations for the mean and variance are given to \( o(1/n) \).

(6) S.C.V. assumed normal (\( N_3 \)).

By \( N_3 \) will be denoted the case where the S.C.V. is assumed normal and the mean and variance of \( v \) are given by \( V \) and

\[ \frac{V^2}{n} \left( V^2 + \frac{1}{4} \right) \]

respectively.
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IV. COMPARISONS BETWEEN THE APPROXIMATIONS

For each of these approximations, a table of percentage points, similar to the exact table of Iglewicz (1967) was computed and the results are given in Table I. The results reveal that the approximations are reasonably good, even for moderate values of \( n \) and relatively large values of \( V \). For moderate and large values of \( n \) the \( P \) approximation seems to give the best results. Despite this, it would likely not be used in practice since it is so cumbersome. Of the other approximations, McKay’s can certainly be recommended on the basis of both accuracy and simplicity. Hald’s approximation does not seem to be any simpler to use than McKay’s and, in general, the results do not seem to be as good. Surprisingly, \( N3 \) seems to perform much better than either \( N1 \) or \( N2 \). In fact, for large values of \( n \), \( N3 \) gives exceptional results throughout the considered range.

REFERENCES


