On Sequenial Tests of the Binomial Distribution

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In this paper, we consider the detailed characteristics of the standard sequential test of a simple hypothesis versus a single alternative for the case of the binomial distribution. A method is developed which permits evaluation of the associated probabilities of acceptance and/or rejection of the null hypothesis at each stage. From this information, the quantitative features of the test (Operating Characteristic and test termination probability) are completely determined. Curves are presented to facilitate assessment of the sequential procedure in practical cases. A sequential test is compared with a fixed sample size test having approximately the same Operating Characteristic. Wald's approximation for the average sample number is compared with exact values.

1. INTRODUCTION

Consider an experiment which can result in success with probability p and failure with probability $1 - p$. The problem to be considered is that of testing the (null) hypothesis,

$$H_0 : p = p_0 , \quad \text{against the alternative}$$

$$H_1 : p = p_1 < p_0$$

A simple test based on the observation of a fixed number of independent samples, N, can be developed as follows: Let α be the magnitude of the Type I error (probability that H_0 will be rejected when true) that can be tolerated, and choose k to most nearly satisfy

$$L(p_0) = E(N, k, p_0) = \sum_{i=k}^{N} \binom{N}{i} p_0^i (1 - p_0)^{N-i} \approx 1 - \alpha$$

H_0 will be accepted if there are at least k out of N successes and rejected (H_1 accepted) otherwise.

The function $L(p)$ is the Operating Characteristic (O.C.) of the test, the probability of accepting H_0 when the true parameter is p. In particular, $L(p_1) = \beta$, the probability of accepting H_0 when H_1 is true (Type II error).

The work of Wald [1] has indicated that some economy in the average number of samples required can be effected by the use of sequential methods. However, it is well known, e.g. Wetherill [2, pg. 21], that the distribution of the sample size typically has a large variance. Since a general, complete, and precise descrip-

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tion of the distributional properties of test sample size is not achievable by the methods of Wald, this topic has received attention.

It is possible to obtain an exact formulation of the properties of the sequential test procedure in cases where the number of possible outcomes of an experiment (or sampling) at any stage is finite.

One example of such a case is the treatment of the binomial distribution as developed by Burman [3]. By formulating an integer approximation to the sequential procedure, he derived and solved difference equations which result in expressions for the O.C. curve, the average sample size, and the variance of the sample size. His formulas are, however, difficult to evaluate numerically.

Polya [4] has studied the case where the slope, s, of the Wald procedure is a rational number. For this case, he has developed recursion methods which permit a complete analysis of the problem. Robinson [5] has performed calculations by Polya's formulae and compared the results with Wald's approximations for eight selected cases. However, even when s is rational, the computational effort is substantial in many cases of interest.

Lindley and Barnett [6] develop an exact analysis for a sequential sampling plan significantly different from that of Wald.

An approach of greater similarity to that developed in this paper is that taken by Aroian [7], in his treatment of the truncated exponential distribution. By developing recursion formulas for probabilities on a grid network, he obtains exact expressions for the O.C. curve and the average time to test termination.

In a recent paper [8], the same author has described a direct method for determining the significant quantitative properties of sequential tests. For a Wald sequential test of a simple hypothesis, the procedure involves the computation of successive convolutions. He has applied this method to truncated sequential tests of the mean of a normal distribution and to a binomial test. For the latter case, he obtains results similar to those in [9].

The literature on sequential analysis is quite extensive, as may be seen from the bibliographies of Johnson [10] and Jackson [11].

In the procedure developed below, we are concerned with non-truncated sequential tests of the binomial distribution, and our objective is to obtain the exact distribution of sample numbers (test stages) required to terminate the test, together with the identification of acceptance and rejection stages. From the distribution, we obtain the mean (ASN), the O.C. function, percentiles, and various other parameters which provide an assessment of specific proposed test designs. This is accomplished by development of a recursion algorithm which can be readily applied to a large class of problems characterized by the binomial distribution, and which requires only modest computing capability.

Numerical results based on the algorithm are presented, offering comparison with the equivalent fixed sample test and Wald's approximate expression for the average sample number [1]. Numerical results to aid in the design of sequential tests are tabulated.

2. Basic Theory of the Method

Following Wald [1], consider a sequential test procedure in which x_1 , x_2 , $x_3 \cdots x_m$, $\cdots$ each represent a success or a failure in successive outcomes of

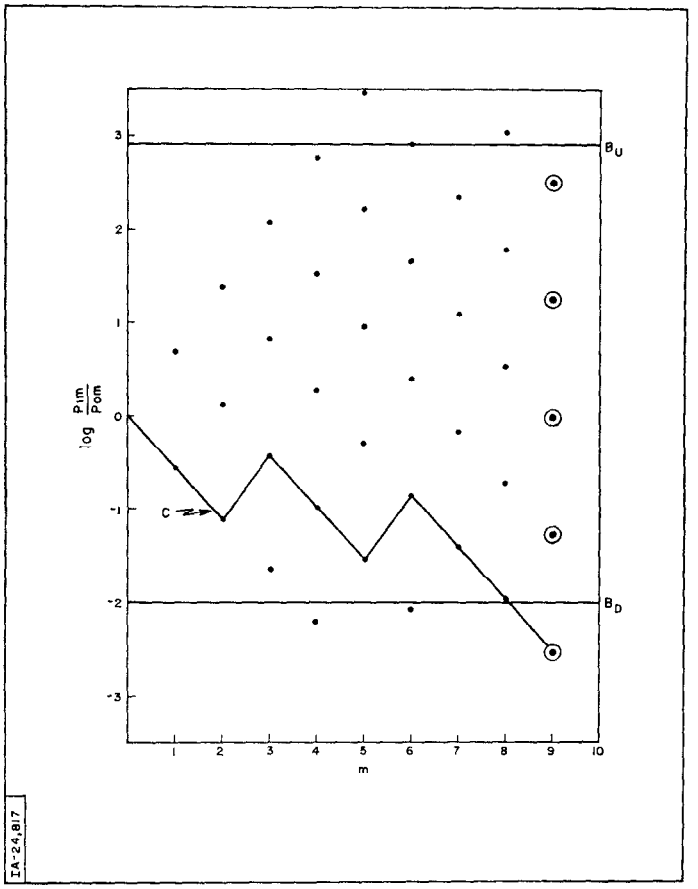


FIGURE 1—Test Diagram

sequential trials. If x_1 is a success, we set $\log (p_{1,1}/p_{0,1}) = \log p_1/p_0 = -S_D$, while if x_1 is a failure, we set $\log (p_{1,1}/p_{0,1}) = \log [(1 - p_1)/(1 - p_0)] = S_U$. For the mth trial, we set $\log (p_{1,m}/p_{0,m}) = \log (p_{1,m-1}/p_{0,m-1}) + S_U$ for a failure or $\log (p_{1,m-1}/p_{0,m-1}) - S_D$ for a success.

A graphical representation of a hypothetical test sequence is depicted in Figure 1, where the sample size, m, is the abscissa, and $\log (p_{1m}/p_{0m})$ is the ordinate (for convenience we abbreviate $p_{1,m}$ to p_{1m}, etc.). Horizontal lines are shown at $B_U = \log [(1 - \beta)/\alpha)]$ and $B_D = \log [\beta/(1 - \alpha)]$. A particular sample sequence can be represented in the test diagram by a step function with discontinuities of either S_U or $-S_D$ at each m. The first time that this discontinuous line intersects the lines B_U or B_D, the test is terminated with rejection or acceptance of the null hypothesis respectively.

Figure 1 shows a test diagram for the parameter values $p_0 = 0.7$, $\alpha = .048$, $p_1 = 0.4$, and $\beta = .127$. These particular values of α and β were selected to coincide with those of a fixed sample size test with $N = 20$, in particular $E(20, 11, 0.7) = 0.952 = 1 - \alpha$ and $E(20, 11, 0.4) = 0.127 = \beta$.

Suppose the test has not terminated prior to step m. Then, the value of $\log p_{1,m}/p_{0,m}$ at step m must assume one of the values $L^{(m)}_{n_1,n_2}$, where

$$L^{(m)}_{n_1,n_2} = n_1 S_U - n_2 S_D \,, \tag{1}$$

where n_1 and n_2 are such that

$$\left.\begin{aligned} &n_1 + n_2 = m; \quad n_1 \geq 0; \quad n_2 \geq 0 \\ \text{and} \quad &B_D - S_D < L^{(m)}_{n_1,n_2} < B_U + S_U \end{aligned}\right\} \tag{2}$$

In Figure 1, these possible states are illustrated for $m = 9$ by the circled points; $L^{(9)}_{6,3}$; $L^{(9)}_{5,4}$; $L^{(9)}_{4,5}$; $L^{(9)}_{3,6}$; and $L^{(9)}_{2,7}$.

We can associate with each state $L^{(m)}_{n_1,n_2}$ a probability $p^{(m)}_{n_1,n_2}$, that $\log p_{1m}/p_{0m} = L^{(m)}_{n_1,n_2}$ after m samples have been processed. This probability is the sum of the probability associated with each sample sequence for which $B_D < \log p_{1i}/p_{0i} < B_U$ for $j = 1, 2, \cdots, m-1$ and $\log p_{1m}/p_{0m} = L^{(m)}_{n_1,n_2}$. It is easy to show that

$$\left.\begin{aligned} &p^{(m)}_{n_1,n_2} = \epsilon_1(1-p)p^{(m-1)}_{n_1-1,n_2} + \epsilon_2 p\, p^{(m-1)}_{n_1,n_2-1} \\ \text{where} \quad &\epsilon_1 = \begin{cases} 1: & L^{(m-1)}_{n_1-1,n_2} > B_D \\ 0: & L^{(m-1)}_{n_1-1,n_2} \leq B_D \end{cases} \\ &\epsilon_2 = \begin{cases} 1: & L^{(n-1)}_{n_1,n_2-1} < B_U \\ 0: & L^{(m-1)}_{n_1,n_2-1} \geq B_U \end{cases} \end{aligned}\right\} \tag{3}$$

Starting from the relation $p^{(1)}_{1,0} = 1 - p$ and $p^{(1)}_{0,1} = p$, we can, by recursion, obtain $p^{(m)}_{n_1,n_2}$ for any m and for n_1 and n_2 satisfying equation (2).
If for a given m,

$$IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] > \frac{(m-1)S_U - B_U}{S_U + S_D} \tag{4a}$$

where IP represents the integer part, there is a rejection at the mth stage with

$$n_1 = m - IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] \tag{4b}$$

For a proof of equation (4), see the Appendix. If

$$IP\left[\frac{mS_D + B_D}{S_U + S_D}\right] > \frac{(m-1)S_D + B_D}{S_U + S_D} \tag{5a}$$

there is an acceptance at the mth stage with

$$n_1 = IP\left[\frac{mS_D + B_D}{S_U + S_D}\right] \tag{5b}$$

The sum over all m, satisfying (5a), of the probabilities $p^{(m)}_{n_1,m-n_1}$, where the values of n_1 for each such m are given by (5b) is the Operating Characteristic. The probability of termination at stage m is the sum over all n_1 satisfying (4) or (5) of the probabilities $p^{(m)}_{n_1,m-n_1}$; the aggregate of these termination prob-

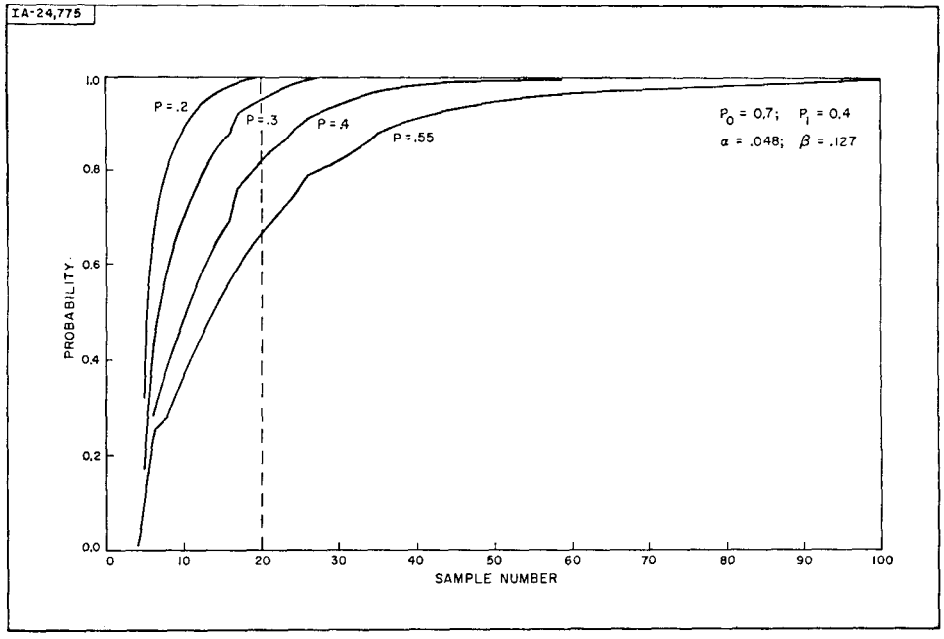


FIGURE 2—Cumulative Termination Probability vs Sample Size

abilities constitutes the probability distribution function of the test termination time and the ASN is the expected value of this random variable.

Although it is possible to proceed further analytically, equations (1)–(5) constitute a suitable algorithm for a computer solution.

3. Results and Discussion

The algorithm described above has been implemented to yield distributions of the sample size. The run terminates as soon as the cumulative probability of termination exceeds 0.999.

An objective of this study was to compare sequential tests against standard tests with fixed sample sizes. The case selected for comparison is that depicted in Figure 1 ($p_0 = 0.7$, $\alpha = .048$, $p_1 = 0.4$, $\beta = .127$), corresponding to a fixed sample size with $N = 20$ and $k = 11$.

The basic information obtained is depicted in the plots of Cumulative Termination Probability, shown as Figure 2. An alternative means of depicting this information is by means of Termination Probability Contours, shown in Figure 3. This curve is obtained directly from the corresponding Figure 2 and shows the number of samples required for 50%, 80%, 95%, and 99.9% probability of termination as a function of the true (unknown) probability, p. Because of the approximate symmetry in the Termination Contours, it suffices to show only values of $p \leq 0.55$ in Figure 2.

In Figure 4, we have shown both the O.C. curve and the probability of terminating before the corresponding fixed sample size case. The crosses on Figure 4 show the corresponding points on the O.C. curve for the fixed sample size test.

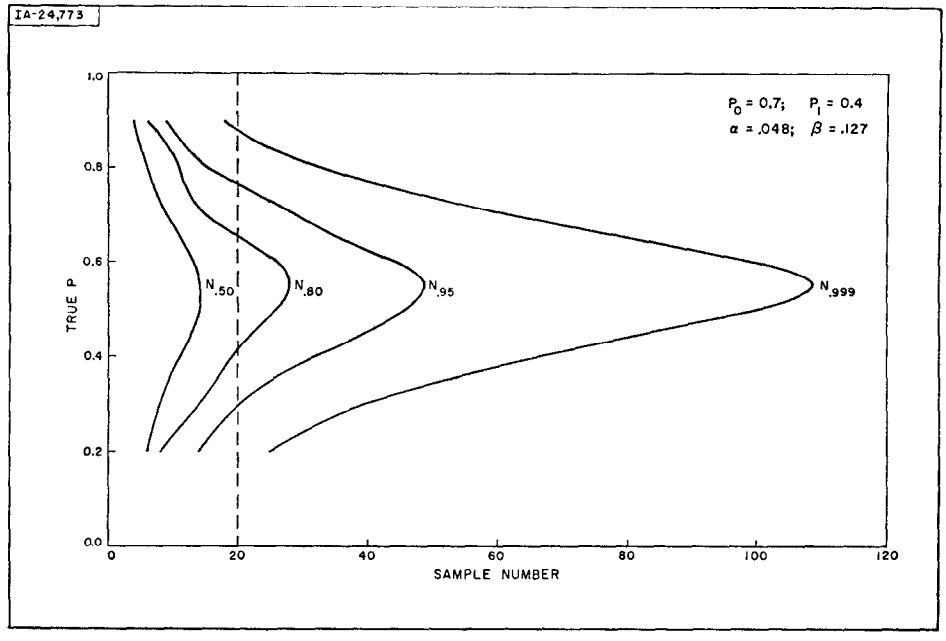


FIGURE 3—Termination Probability Contours

It is clear from Figures 2, 3, and 4 that for a range of values of p in the vicinity of $p = \frac{1}{2}(p_0 + p_1)$, the sequential test will not be preferable to the fixed sample size test, unless a suitable rule for termination of such cases is employed ([1], Section 3.8).

Figure 5 shows the discrete probability function for the number of samples required to terminate the test. The data is shown for $p = p_1 = 0.4$.

There are a number of factors that determine the form of the termination probability distribution. At a given stage, there is a probability of acceptance, of rejection, of neither, or of both. For example, in the case shown, trials 5, 6, 8, 10, 12, 14, 16, 17, 19, $\cdots$ may result in rejection, trials 4, 6, 9, 11, 13, 15, $\cdots$ may result in acceptance, trials 6, 26, 35, $\cdots$ may result in either, and trials 7, 27, 36, $\cdots$ can result in neither.

For neighboring stages, the probabilities associated with rejection are considerably larger than those associated with acceptance. For the case under consideration, it is seen from equations (4a) and (5a) that two successive rejection trials can have at most one non-rejection trial between them and also that two successive acceptance trials can have either one or two non-acceptance trials between them.

It is also of interest to assess the accuracy of Wald's approximate expression for the ASN [1, page 99]. In Figure 6 we show the mean sample number as computed by the program, and also Wald's approximate expression.

Figure 6 indicates that Wald's approximation may be quite misleading in some cases—it predicts a maximum ASN approximately 20% less than the exact calculation (there is some evidence that the approximation is better in large sample cases). However, in view of the very large spread in sample number,

we do not believe the ASN function to be a particularly useful measure of the effort required to conduct a sequential test. The Termination Probability Contours appear to be a more useful guide to the amount of sampling required.

4. Summary Data For Test Design

Table I presents pertinent summary data for a group of parameter values (p_0 , p_1 , α, β) which were chosen to have a fairly general range of applicability. For each set of parameters, we tabulate $L(p)$ (the O.C. function), the average sample number (as computed by the program), and the sample number for which there is a 95 percent probability of terminating the test.

Because of the symmetry of the problem when $\alpha = \beta$ it is redundant to tabulate the quantities for $p_0 < 0.5$. Thus, in designing tests for such cases ($p_0 < 0.5$), one enters the appropriate table with $p' = 1 - p$ and employs the relations

$$N_{.95}(p') = N_{.95}(p) \qquad \text{ASN } (p') = \text{ASN } (p) \qquad L(p') = 1 - L(p)$$

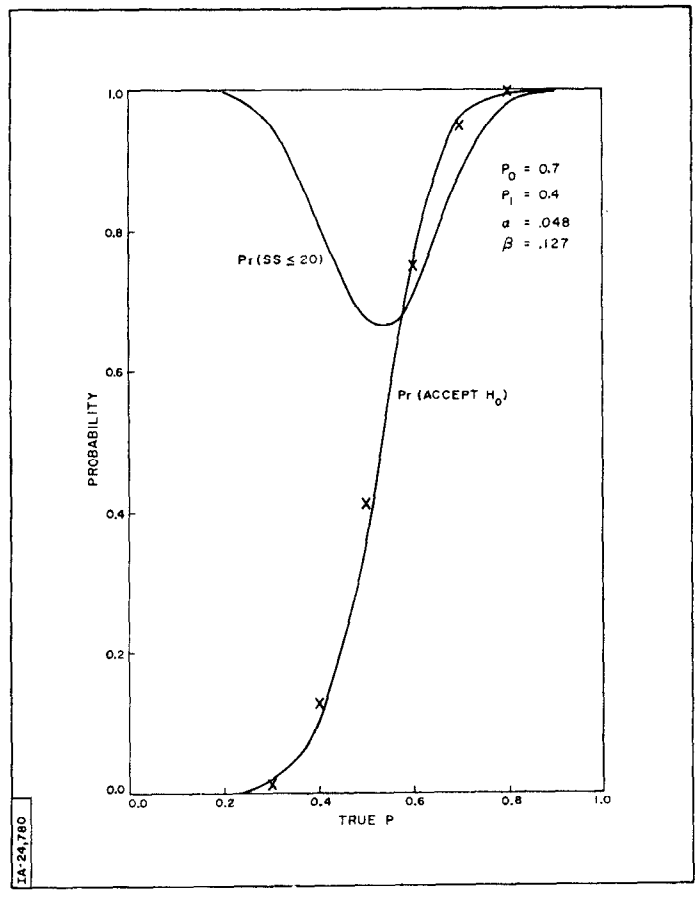


Figure 4—Test Characteristics

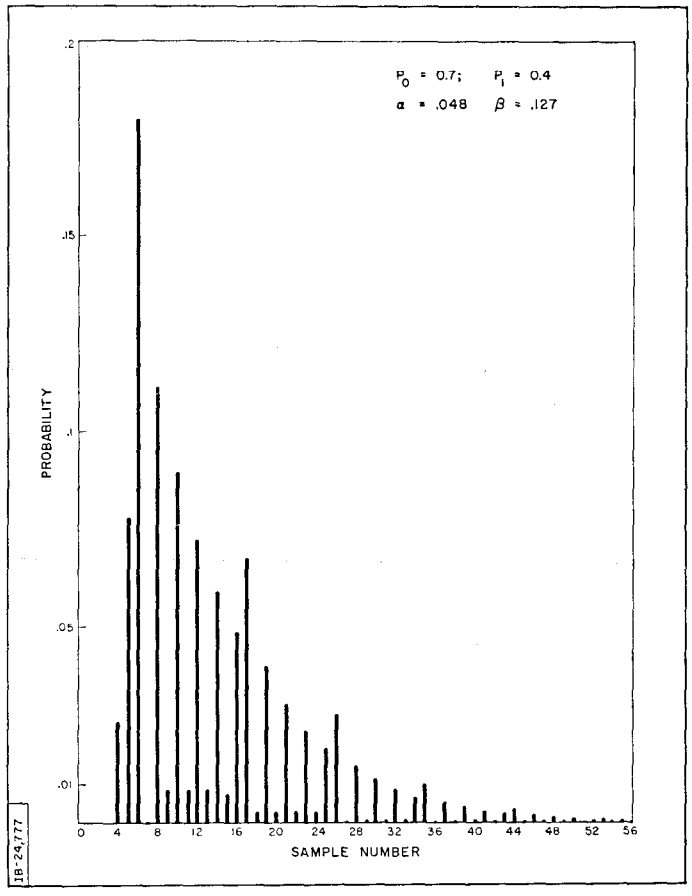


FIGURE 5—Termination Probability Density (true $p = 0.4$)

SUMMARY

A method has been developed for determining the exact statistical properties of the binomial sequential test. This theory has been applied to a comparison of a sequential test with an equivalent fixed sample test and with Wald's approximate expression for the average sample number. In addition, the statistical properties of sequential tests for a number of parameters of general interest for test design, are tabulated.

APPENDIX-THE REJECTION STATES

A sequential test, as defined in Section 2, terminates at the mth sample with rejection of the null hypothesis if integers n_1 and n_2 are such that the following inequality is satisfied:

$$B_U \leq L^{(m)}_{n_1,n_2} < B_U + S_U \,. \qquad \text{A-1}$$

Writing $L^{(m)}_{n_1,n_2}$ in the form

$$L^{(m)}_{n_1,n_2} = (m - n_2)S_U - n_2 S_D ,$$

the inequality A-1 can be written as an inequality for n_2 in the form:

$$\frac{(m-1)S_U - B_U}{S_U + S_D} < n_2 \leq \frac{mS_U - B_U}{S_U + S_D} \qquad \text{A-2}$$

To select a non-negative integer n_2 satisfying A-2, it is necessary that the upper limit be non-negative. That is, there can be rejection states only for

$$m \geq B_U/S_U \qquad \text{A-3}$$

Each value of m defines an interval I_m on the real line defined by equation A-2. Each such interval is of length $S_U/(S_U + S_D) < 1$. Thus, a given interval can contain at most one integer. It is clear that if an integer lies within the interval I_m ,

$$IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] > \frac{(m-1)S_U - B_U}{S_U + S_D} \qquad \text{A-4}$$

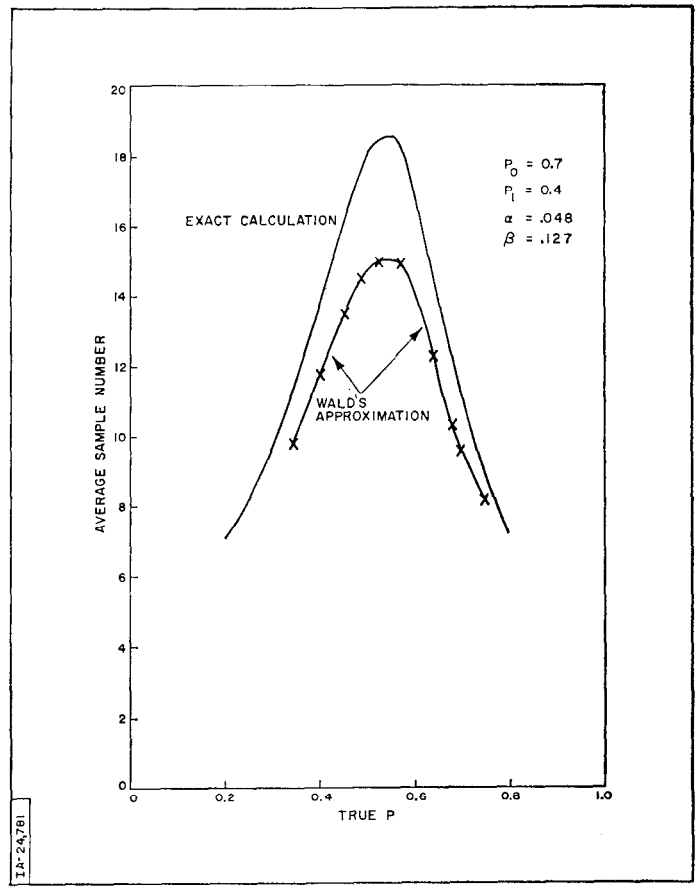


FIGURE 6—Comparison of Average Sample Number

TABLE I

Test Summary Data

$p_0 = 0.9$, $p_1 = 0.7$

p	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	3.0	3	0.000	3.0	3
0.1	.000	3.4	4	.000	3.3	5
0.2	.000	4.0	7	.000	3.7	6
0.3	.000	4.8	9	.000	4.3	7
0.4	.000	6.0	12	.000	5.1	10
0.5	.000	8.0	17	.002	6.6	15
0.55	.001	9.5	20	.006	7.6	17
0.6	.004	11.6	27	.015	9.1	22
0.65	.013	14.9	36	.038	11.3	27
0.7	.045	20.0	51	.093	14.3	36
0.75	.147	27.6	73	.216	18.1	47
0.8	.416	36.2	94	.449	21.5	55
0.85	.786	34.6	87	.742	21.3	52
0.9	.966	24.4	55	.933	17.0	36
0.95	.997	16.6	28	.993	12.3	25
1.00	1.000	12.0	12	1.000	9.0	9

$p_0 = 0.8$, $p_1 = 0.6$

p	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	5.0	5	0.000	4.0	4
0.1	.000	5.7	8	.000	4.4	6
0.2	.000	6.7	10	.000	5.1	8
0.3	.000	8.3	15	.000	6.2	12
0.4	.000	10.8	20	.001	8.2	16
0.45	.000	12.8	26	.004	9.7	22
0.5	.002	15.8	34	.011	11.8	28
0.55	.011	20.4	47	.032	14.9	36
0.6	.044	28.0	68	.088	19.3	49
0.65	.160	39.4	102	.224	24.7	65
0.7	.458	49.1	129	.472	28.5	74
0.75	.806	43.9	110	.750	26.9	70
0.8	.960	31.0	72	.920	21.4	50
0.85	.994	21.8	45	.982	16.1	35
0.9	.999	16.4	28	.997	12.3	22
1.00	1.000	11.0	11	1.000	8.0	8

$p_0 = 0.7$, $p_1 = 0.5$

p	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	6.0	6	0.000	5.0	5
0.1	.000	7.4	11	.000	5.7	8
0.2	.000	9.4	16	.000	7.0	11
0.3	.000	12.5	23	.000	9.2	18
0.35	.000	15.0	29	.002	11.0	23
0.4	.002	18.5	38	.008	13.6	31
0.45	.010	24.0	54	.027	17.3	41
0.5	.044	33.1	80	.084	22.7	57
0.55	.170	46.6	119	.227	29.0	75
0.6	.483	56.4	148	.484	32.9	85
0.65	.817	48.6	125	.755	30.3	78
0.7	.959	34.3	82	.914	24.0	57
0.75	.992	24.3	52	.976	18.3	40
0.8	.998	18.4	34	.994	14.2	30
0.9	.999	12.2	19	.999	9.5	15
1.00	1.000	9.0	9	1.000	7.0	7

$p_0 = 0.6$, $p_1 = 0.4$

p	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	8.0	8	0.000	6.0	6
0.1	.000	10.0	14	.000	7.5	12
0.2	.000	13.3	22	.000	10.0	18
0.25	.000	15.9	30	.001	11.9	24
0.3	.001	20.0	40	.006	14.7	32
0.35	.007	26.2	58	.024	19.0	44
0.4	.038	36.8	88	.081	25.0	62
0.45	.167	52.9	136	.231	32.1	82
0.5	.500	63.6	168	.500	35.8	94
0.55	.832	52.9	136	.769	32.1	82
0.6	.962	36.8	88	.918	25.0	62
0.65	.992	26.2	58	.975	19.0	44
0.7	.998	20.0	40	.993	14.7	32
0.8	.999	13.3	22	.999	10.0	18
0.9	.999	10.0	14	.999	7.5	12
1.00	1.000	8.0	8	1.000	6.0	6

where IP denotes integer part. It also follows that the integer n_2 is given by

$$n_2 = IP\left[\frac{mS_U - B_U}{S_U + S_D}\right]$$

with

$$n_1 = m - IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] \qquad \text{A-5}$$

The intervals I_k, $k \geq B_U/S_U$, $k = k_1, k_1 + 1, k_1 + 2, \cdots$ contain all points X on the real line for which

$$\frac{(k_1 - 1)S_U - B_U}{S_U + S_D} < X < \infty$$

and each such point X lies in only one interval since the intervals are disjoint. Suppose that X lies in I_k. As X increases, it passes into I_{k+1}, then I_{k+2} and so on. For the mth interval, let

$$IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] = n_m$$

(The quantity n_m may or may not lie in I_m—in the former case, $n_m = n_2$ and there is a rejection at state m). There are $n_m + 1$ non-negative integers $n_k \leq n_m$ and each of these integers lies in an interval I_k for $k \leq m$. Thus there are

$$IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] + 1$$

rejection states for m or less samples.

TABLE I

Test Summary Data (Continued)

$p_0 = 0.9$, $p_1 = 0.8$

	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
p	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	5.0	5	0.000	4.0	4
0.1	.000	5.5	7	.000	4.4	6
0.2	.000	6.3	9	.000	5.0	7
0.3	.000	7.3	12	.000	5.8	10
0.4	.000	8.8	15	.000	7.0	13
0.5	.000	11.2	20	.000	8.8	17
0.6	.000	15.5	30	.000	12.2	26
0.7	.000	25.5	55	.003	20.0	45
0.75	.004	37.3	85	.016	28.2	69
0.8	.048	65.1	164	.097	45.1	116
0.825	.160	91.0	239	.226	57.4	150
0.85	.443	115.2	304	.465	66.9	176
0.875	.797	105.5	266	.745	64.3	164
0.9	.960	74.6	170	.922	51.3	122
0.95	.999	38.3	67	.998	29.0	54
1.0	1.000	25.0	25	1.000	19.0	19

$p_0 = 0.8$, $p_1 = 0.7$

	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
p	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	8.0	8	0.000	6.0	6
0.1	.000	9.0	12	.000	6.8	9
0.2	.000	10.4	14	.000	8.1	11
0.3	.000	12.7	20	.000	9.9	15
0.4	.000	16.3	26	.000	12.7	22
0.5	.000	22.7	41	.000	17.7	34
0.6	.000	37.7	77	.002	28.9	62
0.65	.003	55.9	122	.014	41.6	96
0.7	.047	99.6	244	.096	67.6	172
0.725	.170	139.7	361	.236	85.7	225
0.75	.473	171.0	450	.484	97.3	257
0.775	.809	150.0	382	.752	90.5	235
0.8	.956	106.4	253	.913	71.8	174
0.85	.998	57.2	111	.994	42.1	90
0.9	.999	38.2	63	.999	28.3	49
1.0	1.000	23.0	23	1.000	17.0	17

$p_0 = 0.7$, $p_1 = 0.6$

	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
p	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	11.0	11	0.000	8.0	8
0.1	.000	12.7	17	.000	9.6	13
0.2	.000	15.3	21	.000	11.7	17
0.3	.000	19.7	31	.000	14.9	25
0.4	.000	27.6	48	.000	20.8	37
0.5	.000	45.7	89	0.001	34.4	73
0.55	.003	67.9	146	.012	50.3	116
0.6	.047	122.3	299	.094	82.8	209
0.625	.175	171.8	444	.239	104.9	274
0.65	.486	206.7	545	.492	117.5	310
0.675	.814	178.0	456	.754	107.8	278
0.7	.954	126.4	303	.909	85.4	213
0.75	.997	68.6	140	.991	50.8	109
0.8	.999	45.7	80	.999	34.3	63
0.9	.999	27.2	40	.999	20.6	32
1.0	1.000	20.0	20	1.000	15.0	15

$p_0 = 0.6$, $p_1 = 0.5$

	$\alpha = \beta = 0.05$			$\alpha = \beta = 0.10$		
p	L(p)	ASN	$N_{.95}$	L(p)	ASN	$N_{.95}$
0.0	0.000	14.0	14	0.000	13.0	13
0.1	.000	16.9	21	.000	12.5	18
0.2	.000	21.5	32	.000	16.2	25
0.3	.000	30.0	48	.000	22.6	39
0.4	.000	49.8	94	.001	37.5	76
0.45	.002	74.1	155	.010	55.1	124
0.5	.046	134.5	326	.093	91.0	228
0.525	.179	189.0	488	.241	115.1	299
0.55	.495	224.7	593	.497	127.6	337
0.575	.817	191.1	492	.756	116.0	301
0.6	.953	135.8	328	.908	91.8	228
0.65	.997	74.3	155	.990	55.1	122
0.7	.999	49.7	92	.998	37.4	75
0.8	.999	29.9	48	.999	22.5	37
0.9	.999	21.5	30	.999	16.3	21
1.0	1.000	17.0	17	1.000	13.0	13

Corresponding results for acceptance terminations (equations 5a and 5b) can be proved in a similar fashion.

The total number of termination states for m or less samples is given by:

$$IP\left[\frac{mS_U - B_U}{S_U + S_D}\right] + IP\left[\frac{mS_D + B_D}{S_U + S_D}\right] + 2.$$

References

[1] Wald, A. (1947). *Sequential Analysis*, John Wiley and Sons, New York.

[2] Wetherill, G. B. (1966). *Sequential Methods in Statistics*, John Wiley and Sons.

[3] Burman, J. P. (1946). Sequential Sampling Formulae for a Binomial Population. *J. of the Royal Stat. Soc.*, Supplement to Vol. 8.

[4] Polya, G. (1948). Exact Formulas in the Sequential Analysis of Attributes, *University of California Publications in Mathematics, New Series 1*, p. 229–239.

[5] Robinson, J. (1948). A Note on Exact Sequential Analysis. *University of California Publications in Mathematics, New Series 1*, p. 241–246.

[6] Lindley, D. V. and Barnett, B. N. (1965). Sequential Sampling: Two Decision Problems with Linear Losses for Binomial and Normal Random Variables. *Biometrika* (52) p. 507–532.

[7] Aroian, L. A. (1963). Exact Truncated Sequential Tests for the Exponential Density Function, *Ninth National Symposium on Reliability and Quality Control;* Inst. of Radio Engineers.

[8] Aroian, L. A. (1968). Sequential Analysis: Direct Method. *Technometrics 10*, 125–132.

[9] Lloyd, D. and Lipow, M. (1962). *Reliability: Management, Methods and Mathematics.* Prentice-Hall.

[10] Johnson, N. L. (1962). Sequential Analysis: A Survey. *Journal of the Royal Statistical Society, Series A, 124*, p. 372–411.

[11] Jackson, J. E. (1960). Bibliography on Sequential Analysis. *Journal of American Statistical Assoc. 55*, p. 561–580.