Sample Size For Tolerance Limits on a Normal Distribution*

G. DAVID FAULKENBERRY** AND JAMES C. DALY
Department of Statistics, Oregon State University, Corvallis, Oregon

Sample size tables are given for tolerance limits on a normal distribution. Wald-Wolfowitz two-sided limits and one-sided limits are considered. The criterion used for determining sample size is as follows: For a tolerance limit such that Pr (coverage $\geq P$) = $\gamma$, choose $P' > P$ and $\delta$ (small) and require Pr (coverage $\geq P'$), $\leq \delta$. Five levels of $P'$, three levels of $\gamma$, three levels of $P'$, and three levels of $\delta$ are used in the tables. The tables are given for the common case where the degrees of freedom for the $\chi^2$ is one less than the sample size, but it is shown how to use the tables for other cases which occur in simple linear regression and some experimental designs. Examples are given to illustrate the use of the tables.

1. INTRODUCTION

Let $X_1, \cdots, X_N$ represent a random sample from a distribution with density $f(x; \theta)$. We are interested in tolerance limits $L_1 = L_1(X_1, \cdots, X_N)$ and $L_2 = L_2(X_1, \cdots, X_N)$ such that

$$\Pr \left\{ \int_{-\infty}^{L_1} f(x; \theta) \, dx \geq P \right\} = \gamma.$$  (1)

Wald and Wolfowitz [9] derived limits of this type for sampling from a normal distribution. Their solution was of the form $L_1 = \bar{x} - Ks$, $L_2 = \bar{x} + Ks$, where

$$\bar{x} = \left( \frac{1}{N} \right) \sum_{i=1}^{N} x_i,$$

$$s = \left( \sum_{i=1}^{N} (x_i - \bar{x})^2 \right)^{1/2},$$

and

$$K = \left( \frac{N - 1}{\chi^2(N - 1)} \right)^{1/4} r(N^{-1}, P).$$

$\chi^2(N - 1)$ is defined by $\Pr (\chi^2(N - 1) > \chi^2(N - 1)) = \gamma$, and $r(N^{-1}, P)$ is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-t^{2}/2} \, dt = P.$$  (2)

The confidence level, $\gamma$, for such limits is an approximation. That is, with $K$

* Research supported in part by the National Science Foundation, NSF Grant GP-7491.
** Now at Litton Scientific Support Laboratories, Fort Ord, California.

Received Dec. 1968; revised Mar. 1969

813
determined as above, \( \bar{X} \pm KS \) actually corresponds to \( N, P, \gamma'. \) Ellison [2] has shown that \(|\gamma - \gamma'|\) is on the order of \( 1/N. \) In the work that follows it will be understood that the confidence levels are approximations. Wald and Wolfowitz [9] give some calculations which indicate that for some of the common values of \( P \) and \( \gamma \) the approximations are quite good even for small sample sizes. For results on the calculations of \( K \) see Weissberg and Beatty [11] and Gardiner and Hull [5].

As in many statistical problems, the question of sample size arises when one is using tolerance limits. Wilks [12] considered the sample size question in the non-parametric case. In the parametric case Faulkenberry and Weeks [3] suggested that the sample size question be considered as follows: for tolerance limits which satisfy (1), and for \( P' > P, \)

\[
\Pr \left\{ \int_{-\infty}^{x_0} f(x; \theta) \, dx \geq P' \right\}
\]

will generally be a decreasing function of \( N. \) Therefore, the value of (3) can be used as a measure of goodness. In particular, for preassigned \( \delta \) (small), \( N \) could be determined such that (3) is less than or equal to \( \delta. \)

As an illustration of how one might consider sample size in the tolerance limits problem, consider the following: A new hot water tank is to be installed in a plant. It is decided to take some observations on daily hot water usage to determine the appropriate size tank to install. An observation is to be expressed as the size tank which would have been adequate for that day. There are obviously two conflicting goals; (1) to have the tank large enough to provide an adequate supply of hot water most of the time, and (2) to keep the cost down, realizing that the larger the tank the more expensive to buy, install and operate. Suppose it is agreed that a tank will be satisfactory if on 95% of the days it provides enough hot water. This suggests an upper tolerance limit on 95% of the distribution. If the upper limit which we eventually get includes more than 95% of the distribution, then the installed tank will be larger than was agreed would be satisfactory. For example, suppose we consider that \( \sigma \) will be in the neighborhood of 200 gallons. Then if our estimate were at the 97.5% point, the tank would be considered (1.96-1.645) \((200) = 63\) gallons bigger than "satisfactory"; and if it were at the 99% point, it would be (2.32-1.645) \((200) = 135\) gallons bigger than "satisfactory". On the basis of such considerations a \( P' \) larger than .95 could be arrived at with the requirement that the sample size be large enough so that there would be a small probability of the upper tolerance limit exceeding the 100P% point.

In this paper, using the above criterion, tables of sample sizes are determined for two-sided and one-sided tolerance limits for a normal distribution. Some applications are given to illustrate the use of the tables and the techniques by which the tables were obtained.

2. SAMPLE SIZE FOR TWO-SIDED TOLERANCE LIMITS

Let \( f(x; \mu, \sigma) \) be the normal density with unknown mean and variance and let \( K \) and \( r \) be defined as above. Then to within order \( 1/N, \)
TOLERANCE LIMITS ON A NORMAL DISTRIBUTION

\[ P \left( \int_{\mu - K \bar{S}}^{\mu + K \bar{S}} n(x; \mu, \sigma) \, dx \geq P \right) = \gamma. \quad (4) \]

This statement is satisfied for any \( N \geq 2 \). Now for given \( P' > P \) and \( \delta \), we are interested in determining \( N \) such that

\[ P \left( \int_{\mu - K' \bar{S}}^{\mu + K' \bar{S}} n(x; \mu, \sigma) \, dx \geq P' \right) \leq \delta. \quad (5) \]

Using the Wald-Wolfowitz technique, the solution to

\[ P \left( \int_{\mu - K' \bar{S}}^{\mu + K' \bar{S}} n(x; \mu, \sigma) \, dx \geq P' \right) = \delta \]

is

\[ K' = \left( \frac{N - 1}{\chi^2(N - 1)} \right)^{1/4} r(N^{-1}, P'), \]

where

\[ r(N^{-1}, P') \]

is the solution to the equation

\[ \frac{1}{\sqrt{2\pi}} \int_{(N^{-1})^{-1}}^{(N^{-1})^{-1}} e^{-t^2/2} \, dt = P'. \]

Therefore, we see that in order to have both (4) and (5) satisfied, we must choose \( N \) large enough so that \( K \leq K' \). This gives

\[ \left( \frac{N - 1}{\chi^2(N - 1)} \right)^{1/4} r(N^{-1}, P) \leq \left( \frac{N - 1}{\chi^2(N - 1)} \right)^{1/4} r(N^{-1}, P') \]

or

\[ \left[ \frac{r(N^{-1}, P)}{r(N^{-1}, P')} \right]^2 \chi^2(N - 1) \leq \chi^2(N - 1). \quad (6) \]

To solve this inequality for \( N \), Bowker [1] gives

\[ r(N^{-1}, P) = r(0, P) + [r(0, P) - 2r(0, P)]/2N + (3r(0, P) - 2[r(0, P)])^3/24N^2 + 0(1/N^3), \]

where \( r(0, P) \) is the standard normal value exceeded with probability \( (1 - P)/2 \).

The chi-square values can be looked up in the tables of Harter [6] which gives degrees of freedom up to 330, or for degrees of freedom greater than 330 the approximation

\[ \chi^2 \approx 1/2 \left[ Z_a + \sqrt{2n - 1} \right]^2 \]

is appropriate, \( n \) is the degrees of freedom and \( Z_a \) is the standard normal value exceeded with probability \( \alpha \).

The calculations can be simplified by observing that

\[ \frac{r(N^{-1}, P)}{r(N^{-1}, P')} = \frac{r(0, P)}{r(0, P')}, \]

for reasonably large sample size. For example with \( P = .95, P' = .99, \) and
$N = 50$ we have

$$\frac{r(1/\sqrt{50}, .95)}{r(1/\sqrt{50}, .99)} = \frac{1.9794}{2.6012} = .76096$$

while

$$\frac{r(0, .95)}{r(0, .99)} = \frac{1.9600}{2.5758} = .76093.$$  

For the conditions used in Table 1, the smallest sample size encountered was 46. We should also note that for certain values of $P$ and $P'$, $r$ can be obtained from the tables of Weissberg and Beatty [11].

To illustrate how the sample sizes are obtained, suppose a tolerance interval with $P = .95$ and $\gamma = .90$ is desired, and it is decided to impose the conditions $P' = .97$ and $\delta = .10$. We first observe that the ratio

$$\left(\frac{r(0, .95)}{r(0, .97)}\right)^2 = .815806.$$  

We must then satisfy the inequality

$$(815806)x^2_{.95}(N - 1) \leq x^2_{.95}(N - 1).$$

Using the tables of Harter [6] we see that the inequality is first satisfied for $N - 1 = 318$ or $N = 319$. Checking the ratio

$$\frac{r(N - 1, .95)}{r(N - 1, .97)}$$

for $N = 318, 319, 320$ still gives $N = 319$ as the appropriate sample size. All the sample sizes in the table were checked in this manner.

Tables for tolerance limits for a normal distribution generally use $P = .50, .75, .90, .95, \text{and} .99$ with $\gamma = .90, .95, \text{and} .99$. Accordingly we have used these values of $P$ and $\gamma$, and in addition, three values of $P'$ were chosen depending on $P$ (except for $P = .99$ where only two $P'$ values were used) together with $\delta = .10, .05, \text{and} .01$. Thus, each pair $(P, P')$ has nine different sets of confidence coefficients.

**Example:** To illustrate the use of Table I, suppose one wants a tolerance interval for a normal distribution which will contain at least 75% of the distribution with confidence coefficient .90. This means we would take $P = .75$ and $\gamma = .90$. If in addition it is desirable to have a large enough sample size so that the probability of the interval containing more than 80% of the distribution is no more than .05, then we would take $P' = .80$ and $\delta = .05$. Looking under these entries in the table we find $N = 365$.

Therefore, the tolerance interval would be $\bar{x} \pm Ks$, where

$$K = \left(\frac{364}{x^2_{.95}(364)}\right)^{1/4}r(1/\sqrt{365}, .75) = 1.20995.$$  

Normally the $K$ value can be looked up in tables such as Weissberg and Beatty [11] once the sample size is determined.
### Table I

**Sample Sizes for Two-Sided Tolerance Limits for a Normal Distribution**

<table>
<thead>
<tr>
<th>( P' )</th>
<th>( P = .50 )</th>
<th>( P = .75 )</th>
<th>( P = .90 )</th>
<th>( P = .95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( .52 )</td>
<td>( .55 )</td>
<td>( .58 )</td>
<td>( .60 )</td>
</tr>
<tr>
<td>( .90 )</td>
<td>1532</td>
<td>1985</td>
<td>2993</td>
<td>2599</td>
</tr>
<tr>
<td>( .95 )</td>
<td>2068</td>
<td>2523</td>
<td>3646</td>
<td>3422</td>
</tr>
<tr>
<td>( .99 )</td>
<td>3075</td>
<td>3704</td>
<td>5044</td>
<td>5228</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>( P = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( .77 )</td>
</tr>
<tr>
<td>( .90 )</td>
<td>1814</td>
</tr>
<tr>
<td>( .95 )</td>
<td>2376</td>
</tr>
<tr>
<td>( .99 )</td>
<td>3636</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>( P = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( .81 )</td>
</tr>
<tr>
<td>( .90 )</td>
<td>3441</td>
</tr>
<tr>
<td>( .95 )</td>
<td>4502</td>
</tr>
<tr>
<td>( .99 )</td>
<td>6918</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>( P = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( .96 )</td>
</tr>
<tr>
<td>( .90 )</td>
<td>1499</td>
</tr>
<tr>
<td>( .95 )</td>
<td>1965</td>
</tr>
<tr>
<td>( .99 )</td>
<td>3009</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>( P = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( .995 )</td>
</tr>
<tr>
<td>( .90 )</td>
<td>433</td>
</tr>
<tr>
<td>( .95 )</td>
<td>570</td>
</tr>
<tr>
<td>( .99 )</td>
<td>878</td>
</tr>
</tbody>
</table>

3. **Sample Size for One-Sided Tolerance Limits**

The discussion will be in terms of lower tolerance limits, but corresponding upper limits can be obtained by changing the sign of \( K \).

Using the same notation as above, we are interested in finding \( K \) such that

\[
\Pr \left\{ \int_{-\infty}^{x} n(x; \mu, \sigma) \, dx \geq P \right\} = \gamma.
\]
(7) is equivalent to 
\[ \Pr (\bar{X} - KS \leq \mu - Z_{1-\gamma} \sigma) = \gamma, \]
and
\[ \Pr (\bar{X} - KS \leq \mu - Z_{1-\gamma} \sigma) = \Pr \left\{ \frac{(\bar{X} - \mu)N^1\alpha}{\sigma} + Z_{1-\gamma}N^1 \leq KN^1 \right\}. \]

We observe that
\[ \left[ \frac{(\bar{X} - \mu)N^1\alpha}{\sigma} + Z_{1-\gamma}N^1 \right] \]
has a non-central t distribution with \( N - 1 \) degrees of freedom and non-centrality \( Z_{1-\gamma}N^1 \). Therefore we should choose
\[ K = N^{-1}t'(N - 1, Z_{1-\gamma}N^1, 1 - \gamma). \]

For \( K \) chosen in this manner and for \( P' > P \) we have
\[ \Pr \left\{ \int_{\bar{X} - KS}^{+\infty} n(x; \mu, \sigma) \, dx \geq P' \right\} \leq \delta \]
implies \( N \) should be chosen large enough so that
\[ t'(N - 1, N^1Z_{1-\gamma}, 1 - \gamma) \leq t'(N - 1, N^1Z_{1-\gamma}, 1 - \delta). \]

To solve this inequality for \( N \), we used the results of Resnikoff [8], which give
\[ t'(N - 1, N^1\beta, 1 - \alpha) \geq \frac{N^1\beta + \lambda \sqrt{1 + r^2 - \lambda^2/2(N - 1)}}{1 - \lambda^2/2(N - 1)} \]
In this formula \( r = \sqrt{N \beta} / \sqrt{2(N - 1)} \), and \( \lambda \) is a value tabulated by Resnikoff depending on \( 1 - \alpha, \beta, \) and \( N - 1 \). For our problem \( \beta \) will be \( Z_{1-\gamma} \) or \( Z_{1-\gamma} \), and \( \alpha \) will be \( \gamma \) or \( \delta \). An iterative technique was then used to determine the \( N \) for which the inequality was first satisfied. Tables of Owen [7] can be used for some values of \( P, P', \gamma, \) and \( \delta \) and serve as a check on the accuracy. Resnikoff shows that his procedure gives excellent results for \( N - 1 > 9 \).

The table for the one-sided case (Table II) uses the same values of \( P, P', \gamma, \) and \( \delta \) as that for the two-sided case (Table I).

Example: To illustrate the use of Table II suppose we want a lower tolerance limit for a normal distribution which contains 90% of the distribution with probability .90. If in addition we want the probability that it contains more than .95% of the distribution to be at most .10, then we would enter the table with \( P = .90, \gamma = .90, P' = .95, \) and \( \delta = .10 \). For this case the required sample size is 104. \( K \) can then be determined using the formula of Resnikoff [8] or the tables of Owen [7]. For \( N = 104 \) the tolerance limit would be \( \bar{x} - 1.466s \).

4. Case Where \( f \neq N - 1 \)

Wallis [10] points out that if we have any normally distributed variable for whose mean we have a normally distributed estimate with variance \( \sigma^2/N \), and for whose variance we have an estimate independently distributed as \( \sigma^2X^2/f \) for \( f \) degrees of freedom, then the results of Wald and Wolfowitz hold. In this case we will have
### Table II

**Sample Sizes for One-Sided Tolerance Limits for a Normal Distribution**

<table>
<thead>
<tr>
<th>( P' )</th>
<th>.50</th>
<th>.55</th>
<th>.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>.10</td>
<td>.05</td>
<td>.01</td>
</tr>
<tr>
<td>.90</td>
<td>2610</td>
<td>3401</td>
<td>5168</td>
</tr>
<tr>
<td>.95</td>
<td>3402</td>
<td>4298</td>
<td>6261</td>
</tr>
<tr>
<td>.99</td>
<td>5169</td>
<td>6262</td>
<td>8593</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>.75</th>
<th>.80</th>
<th>.85</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>.10</td>
<td>.05</td>
<td>.01</td>
</tr>
<tr>
<td>.90</td>
<td>1993</td>
<td>2592</td>
<td>3928</td>
</tr>
<tr>
<td>.95</td>
<td>2603</td>
<td>4295</td>
<td>4768</td>
</tr>
<tr>
<td>.99</td>
<td>3967</td>
<td>4796</td>
<td>6562</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>.90</th>
<th>.93</th>
<th>.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>.10</td>
<td>.05</td>
<td>.01</td>
</tr>
<tr>
<td>.90</td>
<td>3511</td>
<td>4565</td>
<td>6915</td>
</tr>
<tr>
<td>.95</td>
<td>4586</td>
<td>5782</td>
<td>8396</td>
</tr>
<tr>
<td>.99</td>
<td>6990</td>
<td>8449</td>
<td>11559</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>.96</th>
<th>.97</th>
<th>.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>.10</td>
<td>.05</td>
<td>.01</td>
</tr>
<tr>
<td>.90</td>
<td>1423</td>
<td>1848</td>
<td>2792</td>
</tr>
<tr>
<td>.95</td>
<td>1862</td>
<td>2344</td>
<td>3395</td>
</tr>
<tr>
<td>.99</td>
<td>2844</td>
<td>3432</td>
<td>4684</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P' )</th>
<th>.995</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>.10</td>
<td>.05</td>
</tr>
<tr>
<td>.90</td>
<td>428</td>
<td>553</td>
</tr>
<tr>
<td>.95</td>
<td>562</td>
<td>704</td>
</tr>
<tr>
<td>.99</td>
<td>861</td>
<td>1035</td>
</tr>
</tbody>
</table>

\[
K = \left( \frac{f}{x^2(f)} \right)^* r(N^{-1}, P).
\]

\( N \) is the effective number of observations for the estimate; that is, the number which, when divided into the variance of an observation, gives the variance of the estimate.
Examples: To illustrate the usefulness of this case, we give two examples. First, consider a one-way classification with \( t \) treatments and \( N \) observations per treatment. Assume the errors are independently, normally distributed with common variance. The estimate of a treatment mean will be based on \( N \) observations with variance \( \sigma^2 / N \), and the degrees of freedom for residual mean square will be \( t(N - 1) \). To be explicit, suppose \( t = 5 \) and we are interested in a tolerance interval for a treatment response with \( P = .95 \) and \( \gamma = .90 \). If in addition we want to take enough observations per treatment so that for \( P' = .97 \) we have \( \delta \leq .10 \), then we would need
\[
\left( \frac{t}{\chi^2_{.90}(f)} \right)^{\frac{1}{t}} r(N^{-1}, .95) \leq \left( \frac{t}{\chi^2_{.90}(f)} \right)^{\frac{1}{t}} r(N^{-1}, .97)
\]
or
\[
\left[ \frac{r(N^{-1}, .95)}{r(N^{-1}, .97)} \right] \frac{r^2}{\chi^2_{.90}(f)} \leq \chi^2_{.90}(f).
\]
Using the fact that
\[
\frac{r(N^{-1}, .95)}{r(N^{-1}, .97)} \approx \frac{r(0, .95)}{r(0, .97)}.
\]
We can use Table I to determine the approximate sample size required. Table I gives a "sample size" of 319 for these conditions. Since the table was made for \( f = N - 1 \), we can find \( N \) for this problem by taking \( 5(N - 1) + 1 \geq 319 \) or \( N = 65 \). Thus, 65 observations per treatment would be sufficient. If expression (8) is checked for \( N = 64, 65, 66 \), we find that \( N = 65 \) is appropriate.

As another example, consider the usual simple linear regression problem where \( Y \) is normally distributed with mean \( \alpha + \beta x \) and variance \( \sigma^2 \). Let \( N \) be the number of observations \((x_i, y_i)\). The degrees of freedom, \( f \), for residual mean square is \( N - 2 \), and the effective number of observations for estimating \( E(Y) \) at \( x_0 \) is
\[
\frac{N \sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + N(x_0 - \bar{x})^2}.
\]
This of course depends on the \( x_i's \), but we can consider the case \( x_0 = \bar{x} \) where the effective number of observations will be \( N \). Therefore, if we want a tolerance interval which will contain 95\% of the \( Y \) responses at \( x_0 = \bar{x} \) with probability .90 and is such that it will contain more than 97\% with the probability at most .10, then entering Table I with \( P = .95 \), \( P' = .97 \), \( \gamma = .90 \), and \( \delta = .10 \) gives "sample size" 319. In this case \( f = N - 2 \) so \( (N - 2) + 1 = 319 \) implies \( N = 320 \). For \( x_0 \) in the neighborhood of \( \bar{x} \) these conditions would still be pretty well satisfied since the ratio of the r's is only affected by large changes in the effective number of observations.

In summary, tables are given to determine sample sizes of either one-sided or two-sided tolerance limits with a specified measure of goodness for a normal distribution. The tables are given for the common case where the degrees of freedom for the \( \chi^2 \) is one less than the sample size, but it is shown that they can be used in other cases.
In tolerance limit problems, it is not always clear what values one should choose for $P'$ and $\delta$, the measures of goodness, but the tables could be used as a guide to see what kind of precision to expect for various sample sizes—somewhat like a power curve.

References


