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Confidence Intervals for the Exponential Scale Parameter Using Optimally Selected Order Statistics

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We obtain exact and approximate confidence intervals (tabulated for 90%, 95% and 99%) for the scale parameter, σ, of the exponential distribution in small and large samples. The exact confidence intervals are based on the distributions of the BLUE and ABLE of σ, using k optimally selected order statistics from a random sample of size n and are tabulated for $k = 1(1)5$ and various n. The approximate intervals are based on approximating chi-square distributions. We find that in large samples, the optimal quantiles for the interval estimation of σ are the same as those for the point estimation of σ, for several optimality criteria; or, in other words, these several criteria are equivalent in large samples.

KEY WORDS

Exponential Scale Parameter
Optimum Order Statistics
Quantiles
Confidence Intervals

1. INTRODUCTION

Let $X(1) < X(2) < \cdots < X(n)$ denote the order statistics of a random sample of size n from an exponential population with p.d.f.:

$$e(x; \mu, \sigma) = (1/\sigma) \exp \{(\mu - x)/\sigma\}, \quad x > \mu, \sigma > 0.$$

We wish to find confidence intervals for σ on the basis of k suitably chosen order statistics $X(n_i)$ $(i = 1, 2, \cdots, k)$, where $1 \leq n_1 < n_2 < \cdots < n_k \leq n$ and $1 \leq k \leq n$. These intervals will be obtained from the distributions of σ_1^*, the best linear unbiased estimate (BLUE) of σ, and from those of σ_2^*, the asymptotically best linear estimate (ABLE) of σ.

Point estimation of σ based on selected subsets of the order statistics in small samples has been studied by Harter (1961), Kulldorff (1963b), Saleh (1967), Sarhan, Greenberg and Ogawa (1963), Siddiqui (1963) and Ukita (1955), and the large sample case ($n \rightarrow \infty$) has been treated by Kulldorff (1963a), Ogawa (1960), Saleh (1966) and Saleh and Ali (1966). Interval estimation of σ

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has been studied by Harter (1964) and Kaminsky (1968) for small samples; Ogawa (p. 381 of Sarhan and Greenberg (1962)) and Kaminsky (1968) for large samples.

2. Notation and Some Previous Results

First, it is well known that

$$X(m) = \mu + \sum_{j=1}^{m} (n - j + 1)^{-1} V_j ,$$

$(m = 1, 2, \cdots, n)$, where the V's are mutually independent and distributed as $e(x; 0, \sigma)$.

Now, extending Kulldorff's notation (1963b), let

$$\delta_{ri} = \sum_{j(i)} (n - j + 1)^{-r},$$

where $r = 1, 2, \cdots; i = 1, 2, \cdots, k; \delta_{10}/\delta_{20} = \delta_{1,k+1}/\delta_{2,k+1} = 0$ and the subscript form $u(v)$ will mean throughout that u runs from $n_{v-1} + 1$ to n_v.

If we let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \lambda_{k+1} = 1$, then the population λ_i-quantile of $e(x; 0, 1)$ is $u_i = \ln (1 - \lambda_i)^{-1}$ $(i = 0, 1, \cdots, k)$ and $u_{k+1} = \infty$.

The following results are from Kulldorff (1963a, 1963b) and Ogawa (1960):

(a) If μ is known, the BLUE and ABLE of σ based on the ranks n_i $(i = 1, 2, \cdots, k)$ and $n_0 = 0$ are respectively

$\sigma_1^* = \sigma_1^*(k, n; \mu) = b_0\mu + \sum_{i=1}^k b_i X(n_i)$ and
$\sigma_2^* = \sigma_2^*(k, n; \mu) = B_0\mu + \sum_{i=1}^k B_i X(n_i)$ where
$b_i = (\delta_{1i}/\delta_{2i} - \delta_{1,i+1}/\delta_{2,i+1})/K$ $(i = 0, 1, \cdots, k)$,
$B_0 = u_1/\{(1 - \exp(u_1))K^*\}$,
$B_i = (\Delta_i - \Delta_{i+1})/K^*$ $(i = 1, 2, \cdots, k)$,
$\Delta_i = (u_i - u_{i-1})/\{\exp(u_i) - \exp(u_{i-1})\}$ $(i = 1, 2, \cdots, k)$,
$\Delta_{k+1} = 0$,
$K = \sum_{i=1}^k \delta_{1i}^2/\delta_{2i}$ and
$K^* = \sum_{i=1}^k (u_i - u_{i-1})^2/\{\exp(u_i) - \exp(u_{i-1})\}$. Also,

$\mathrm{Var}(\sigma_1^*) = \sigma^2/K$, while if n is large and $n_i = [n\lambda_i] + 1$ then we have approximately
$\mathrm{Var}(\sigma_2^*) = \sigma^2/(nK^*)$. It can be shown [4] that
$E(\sigma_2^*) = (\sum_{i=1}^k \Delta_i\delta_{1i}/K^*)\sigma$ and
$\mathrm{Var}(\sigma_2^*) = \{\sum_{i=1}^k \Delta_i^2\delta_{2i}/(K^*)^2\}\sigma^2$, exactly. It will follow from Lemma 1 that σ_2^* is unbiased for σ only in the limit $(n \to \infty)$ and the asymptotic variance formula above is valid.

(b) If μ and σ are both unknown, the BLUE and ABLE of σ, based on the same order statistics as in (a), are respectively

$\sigma_1^* = \sigma_1^*(k, n; \mu_1^*) = \sum_{i=1}^k d_i X(n_i)$ and
$\sigma_2^* = \sigma_2^*(k, n; \mu_2^*) = \sum_{i=1}^k D_i X(n_i)$ where
$d_1 = -\delta_{12}/(L\delta_{22})$,
$d_i = (\delta_{1i}/\delta_{2i} - \delta_{1,i+1}/\delta_{2,i+1})/L$ $(i = 2, 3, \cdots, k)$,

$D_1 = -\Delta_1/L^*$,
$D_i = (\Delta_i - \Delta_{i+1})/L^*$ $(i = 2, \cdots, k)$,
$L = \sum_{i=2}^{k} \delta_{1i}^2/\delta_{2i}$ and
$L^* = \sum_{i=2}^{k} (u_i - u_{i-1})^2/\{\exp(u_i) - \exp(u_{i-1})\}$. Also, the BLUE and ABLE of μ are respectively
$\mu_1^* = X(n_1) - \sigma_1^*(k, n; \mu_1^*)\delta_{11}$ and
$\mu_2^* = X(n_1) - \sigma_2^*(k, n; \mu_2^*)u_1$ while
$\text{Var}(\sigma_1^*) = \sigma^2/L$; $\text{Var}(\mu_1^*) = \sigma^2(\delta_{21} + \delta_{11}^2/L)$. (1)

3. Small Samples

If n, the sample size, is small, we will say that the ranks $n_1, \cdots, n_k$ are *optimal* for the *point* or *interval* estimation of σ (or μ) if $\text{Var}(\sigma_1^*)$ (or $\text{Var}(\mu_1^*)$) is a minimum for these ranks over the $\binom{n}{k}$ possible subsamples of size k of the random sample of size n. (Clearly, this is equivalent to maximizing K (or L)). We do this for two reasons: First, as we will see from Theorem 6, various other possible optimality criteria for selecting the ranks are equivalent in large samples to maximizing K (i.e., minimizing $\text{Var}(\sigma_1^*)$). Second, it enables us to use the tables of Harter (1961) and Kulldorff (1963b) where optimal ranks may be found for $k = 1, 2$ in Harter and $k = 3, 4$ and 5 in Kulldorff.

Kulldorff (1963b) also discovered the pleasing fact that if both μ and σ are unknown, then the optimal ranks for estimating them are the same:

Theorem 1: (Kulldorff, 1963b) Let n_i' $(i = 1, \cdots, k-1)$ be the optimal ranks when selecting $k - 1$ order statistics from a sample of size $n - 1$ for the estimation of σ when μ is known; let b_i $(i = 0, \cdots, k-1)$ be the coefficients of the corresponding BLUE $b_0\mu + \sum_{i=1}^{k-1} b_i X(n_i')$; and let σ^2/K' be the variance of this BLUE. Then the variances (1) both attain their minima for $n_1 = 1$ and $n_i = n_{i-1}' + 1$ $(i = 2, \cdots, k)$. Also, $d_i = b_{i-1}$ $(i = 1, \cdots, k)$, $L = K'$, $\text{Var}(\sigma_1^*) = \sigma^2/K'$ and $\text{Var}(\mu_1^*) = \sigma^2(1 + 1/K')/n^2$.

Now, noting that we can write $\sigma_1^*(k, n; \mu) =$

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} b_i(n - j + 1)^{-1} V_j = \sum_{i=1}^{k} \sum_{j(i)} p_{ij} V_j\,, \tag{2}$$

where $p_{ij} = (\delta_{1i}/\delta_{2i})/\{(n - j + 1)K\}$, it can be shown that $\sigma_1^*(k, n; \mu_1^*)$ and $\sigma_1^*(k - 1, n - 1; \mu)$ have the same distribution, as long as $n_1 = 1$, $n_i = n_{i-1}' + 1$ $(i = 2, \cdots, k)$, $\sigma_1^*(k, n; \mu_1^*)$ is based on $n_1, \cdots, n_k$ and $\sigma_1^*(k - 1, n - 1; \mu)$ is based on $n_1', \cdots, n_{k-1}'$ (whether or not these ranks are optimal). From this and from Theorem 1, we can state the particular result:

Theorem 2: For optimal estimation of σ, $\sigma_1^*(k, n; \mu_1^*)$ and $\sigma_1^*(k - 1, n - 1; \mu)$ have the same distribution. (All proofs are omitted but can be found in [4] or by contacting the author).

A useful by-product of this result is that it will not be necessary to generate tables of confidence intervals for σ when μ is unknown, since tables for the case of μ known can be used by entering these tables with k replaced by $k - 1$ and n replaced by $n - 1$ (see Example 2).

The distributions of certain linear combinations of exponential order statistics can be found in Likeš, (1967). We encountered two types of linear combination (2) in compiling Table 2. We now give the distribution of $\sigma_1^*(k, n; \mu)$:

Theorem 3: (a) If in (2), all p_{ij}'s are distinct, then the c.d.f. of $\sigma_1^*(k, n; \mu)/\sigma$ is $F_1(x) = 1 - \sum_{i=1}^{k} \sum_{j(i)} a_{ij} \exp(-x/p_{ij})$, $x > 0$; 0 otherwise, where

$$a_{ij} = \prod_{\substack{r=1 \\ s \neq j}}^{k} \prod_{s(r)} \{p_{ij}/(p_{ij} - p_{rs})\}.$$

(b) If in (2), exactly m of the p_{ij}'s are equal to p while the remaining $n_k - m$ are distinct, then the c.d.f. of $\sigma_1^*(k, n; \mu)/\sigma$ has the form

$$F_1(x) = 1 - \left(\sum_{j=1}^{n_k-m} a_j e^{-x/q_j} + e^{-x/p} \sum_{j=1}^{m} \sum_{i=0}^{m-j} a_{n_k-m+j}(x/p)^i/i!\right),$$

where we have (without loss of generality) rearranged the p_{ij}'s so that the first $n_k - m$ are distinct and these we have renamed q_j ($j = 1, \cdots, n_k - m$). The a's are functions of q_j's and p only.

In Table 2, using the optimal ranks for σ with μ known, found in Harter (1961) and Kulldorff (1963b), we have compiled exact 90%, 95% and 99% confidence intervals for σ, $k = 1(1)5$ and various n. The quantities c_1 and c_2 in the table are defined by $\alpha/2 = P\{\sigma_1^*/\sigma \leq c_2^{-1}(k, n; \alpha)\} = P\{\sigma_1^*/\sigma \geq c_1^{-1}(k, n; \alpha)\}$. Thus, an exact $100(1 - \alpha)\%$ confidence interval for σ, based on the k optimal ranks $n_1, \cdots, n_k$, is

$$(c_1(k, n; \alpha)\sigma_1^*, \quad c_2(k, n; \alpha)\sigma_1^*). \tag{3}$$

We will now compare this interval (3) with the corresponding interval based on the complete sample. We assume that μ is known (and without loss of generality, that it is zero). If the complete sample is used to estimate σ, then the BLUE of σ is simply the sample mean: that is, $\sigma_1^*(n, n; 0) = \bar{X}$. It is well known that $2n\bar{X}/\sigma$ is a chi-square variate with $2n$ degrees of freedom so that $c_1(n, n; \alpha) = 2n/\chi^2_{1-\alpha/2,2n}$ and $c_2(n, n; \alpha) = 2n/\chi^2_{\alpha/2,2n}$. The confidence interval based on $\bar{X}$, corresponding to (3) is

$$(c_1(n, n; \alpha)\bar{X}, \quad c_2(n, n; \alpha)\bar{X}). \tag{4}$$

$\sigma_1^*(k, n; \mu)$ is known to be highly efficient for σ when compared with $\bar{X}$, even for small k. It is natural then to compare the intervals (3) and (4) to see how efficient the interval (3) is relative to (4). One simple way to do this is on the basis of the ratio of expected lengths of the intervals ((4) in the numerator). This ratio is

$$\text{REL}(k, n; \alpha) = \frac{c_2(n, n; \alpha) - c_1(n, n; \alpha)}{c_2(k, n; \alpha) - c_1(k, n; \alpha)}.$$

This quantity has been tabulated in Table 2. We see that the interval (3) is quite efficient relative to the interval (4). The behaviour of REL in large samples is discussed in the next section together with several other criteria for comparing (3) and (4).

4. Large Samples

If we are dealing with large samples, we will base our estimate of σ on the *sample quantiles* $X(n_i)$ $(i = 1, 2, \cdots, k)$ where henceforth, unless otherwise specified, we will have

$$n_i = [n\lambda_i] + 1,$$

where $[\cdot]$ is the greatest integer function. The quantiles u_i $(i = 1, \cdots, k)$ (or equivalently, the λ's) will be called *optimal* for the *point* or *interval* estimation of σ if and only if K^* (or L^* if μ is unknown) is a maximum for these quantiles.

From a result of Kulldorff's (1963a) analogous to Theorem 1, we can state a large sample analogue to Theorem 2:

Theorem 4: For optimal estimation of σ, in large samples, $\sigma_2^*(k, n; \mu_2^*)$ and $\sigma_2^*(k - 1, n - 1; \mu)$ have the same distribution.

Now, the c.d.f. of $\sigma_2^*(k, n; \mu)$, $F_2(x)$, is the same as $F_1(x)$, with p_{ij} replaced by $P_{ij} = \Delta_i/\{(n - j + 1)K^*\}$ (because $\sigma_2^*(k, n; \mu)$ can be written $\sigma_2^*(k, n; \mu) = \sum_{i=1}^{k} \sum_{j(i)} P_{ij}V_j$). We can thus define a $100(1 - \alpha)\%$ confidence interval for σ (based on the optimal quantiles), analogous to (3). It is:

$$(C_1(k, n; \alpha)\sigma_2^*, \quad C_2(k, n; \alpha)\sigma_2^*), \tag{5}$$

where C_1 and C_2 are such that $\alpha/2 = F_2(C_2^{-1}(k, n; \alpha)) = 1 - F_2(C_1^{-1}(k, n; \alpha))$. Although we do not tabulate the interval (5), we discuss its large sample behaviour relative to the small sample ones, and we also discuss approximating it (and the interval (3)) below. First, we state some limit theorems:

Lemma 1: $\delta_{1i} = u_i - u_{i-1} + 0(n^{-1})$ while for $m = 2, 3, \cdots,$

$$n^{m-1}\delta_{mi} = \{e^{(m-1)u_i} - e^{(m-1)u_{i-1}}\}/(m - 1) + 0(n^{-1}).$$

From this lemma we have

Theorem 5: Given the quantiles u_i $(i = 1, \cdots, k)$, then,

(a) $K/n = K^* + 0(n^{-1})$,
(b) $L/n = L^* + 0(n^{-1})$,
(c) while both $\sum_{i=1}^{k} (\delta_{1i}/\delta_{2i}K)^m(n^{m-1}\delta_{mi})$ and $\sum_{i=1}^{k} (\Delta_i/K^*)^m(n^{m-1}\delta_{mi})$ converge to $(K^*)^{-m} \sum_{i=1}^{k} \Delta_i^m(e^{(m-1)u_i} - e^{(m-1)u_{i-1}})/(m - 1)$ as $n \to \infty$

The cumulants of σ_1^*/σ and σ_2^*/σ are easily found to be

$$\omega_m = (m - 1)!\,(K)^{-m} \sum_{i=1}^{k} \delta_{1i}^m \cdot \delta_{mi}/\delta_{2i}^m, \quad \text{and}$$

$$\Omega_m = (m - 1)!(K^*)^{-m} \sum_{i=1}^{k} \Delta_i^m\, \delta_{mi} \qquad (m = 1, 2, \cdots)$$

respectively. The coefficients of skewness and excess (see for example p. 85, Kendall and Stuart, Vol I, 1963) of σ_1^*/σ are:

$$\gamma_1 = 2(K)^{-\frac{3}{2}} \sum_{i=1}^{k} \delta_{1i}^3 \cdot \delta_{3i}/\delta_{2i}^3, \quad \text{and}$$

$$\gamma_2 = 6(K)^{-2} \sum_{i=1}^{k} \delta_{1i}^4 \cdot \delta_{4i}/\delta_{2i}^4 ,$$

respectively, while those of σ_2^*/σ are

$$\Gamma_1 = 2 \sum_{i=1}^{k} \Delta_i^3 \, \delta_{3i}/(\sum_{i=1}^{k} \Delta_i^2 \, \delta_{2i})^{\frac{3}{2}}, \quad \text{and}$$

$$\Gamma_2 = 6 \sum_{i=1}^{k} \Delta_i^4 \, \delta_{4i}/(\sum_{i=1}^{k} \Delta_i^2 \, \delta_{2i})^2 \quad \text{respectively.}$$

These quantities are used later to help justify approximating the distributions of σ_i^*/σ ($i = 1, 2$) with chi-square distributions.

Mosteller (1946) proved under regularity conditions that the k sample quantiles are jointly asymptotically normally distributed, and so it follows that σ_i^*/σ ($i = 1, 2$), properly normalized, converge to the univariate normal. This can be seen directly by noting that the cumulants of $\sqrt{K}(\sigma_1^*/\sigma - 1)$ and $\sqrt{nK^*}(\sigma_2^*/\sigma - 1)$ converge to the cumulants of the standard normal distribution.

Other criteria than those we adopted exist for declaring the ranks or quantiles optimal for estimation of σ. Some of these are (comparing the intervals (3) and (5) to (4), with (4) in the numerator), the maximizing of: the ratio of expected lengths, REL (k, n; α); the ratio of expected squared lengths, RESL (k, n; α), the ratio of variances of the lengths, RVL (k, n; α), and Harter's quantity (1964) of the ratio of the sum of mean squared deviations of the upper and lower confidence bounds from the true value, EFF (k, n; α). In large samples, all of these criteria are equivalent to maximizing K^*. In other words, all of these optimality criteria are equivalent in large samples. This also means that the optimal quantiles for the interval estimation of σ, in large samples, are the same as those for the point estimation of σ. This is particularly useful since the optimal quantiles are tabulated for $k = 1(1)15$ in Sarhan and Greenberg (1962) for example. That these facts are true follows from the next theorem, the proof of which depends on the asymptotic normality of σ_i^*/σ ($i = 1, 2$):

Theorem 6: For k and α given, μ known and σ_i^* ($i = 1, 2$) based on the quantiles u_i ($i = 1, \cdots, k$) (for both (3) and (5) compared with (4)) we have, as $n \rightarrow \infty$,

(a) REL (k, n; α) $\rightarrow \sqrt{K^*}$,
(b) RESL (k, n; α) $\rightarrow K^*$,
(c) RVL (k, n; α) $\rightarrow (K^*)^2$, and
(d) EFF (k, n; α) $\rightarrow K^*$.

5. Approximate Confidence Intervals for σ

A quite satisfactory approximation to the intervals (3) and (5) can be obtained by treating $[2K]\sigma_1^*/\sigma$ and $[2nK^*]\sigma_2^*/\sigma$ as chi-square variates with $[2K]$ and $[2nK^*]$ degrees of freedom respectively. Clearly, this amounts to matching the first two moments of $[2K]\sigma_1^*/\sigma$ and $[2nK^*]\sigma_2^*/\sigma$ with the first two moments of chi-square variates (Harter (1968), suggests using $2K$ rather than $[2K]$

and interpolating in the chi-square tables). The approximate $100(1 - \alpha)\%$ confidence interval corresponding to the intervals (3) and (5) are therefore

$$([2K]\sigma_1^*/\chi^2_{1-\alpha/2,[2K]}\ ,\ [2K]\sigma_1^*/\chi^2_{\alpha/2,[2K]}) \quad \text{and} \tag{6}$$

$$([2nK^*]\sigma_2^*/\chi^2_{1-\alpha/2,[2nK^*]}\ ,\ [2nK^*]\sigma_2^*/\chi^2_{\alpha/2,[2nK^*]}). \tag{7}$$

The asymptotic normality of σ_i^*/σ $(i = 1, 2)$ and χ^2_{df} can be used to show that (given u_i , $i = 1, \cdots, k$) the attained probability content of the intervals (6) and (7) converge to $1 - \alpha$ as n increases, and that the ratio of expected lengths (or any of the other three ratios mentioned in the preceding section) of the intervals (3) to (6) and (5) to (7) converge to one with increasing n. A further comparison can be based on the ratios of coefficients of skewness and excess. For χ^2_{df} , these coefficients are $\xi_1(df) = \sqrt{8/df}$ and $\xi_2(df) = 12/df$. From Theorem 5c we easily see that $\gamma_i/\xi_i([2K])$ and $\Gamma_i/\xi_i([2nK^*])$ converge to $Q_i(k)$ $(i = 1, 2)$ as $n \to \infty$, where

$$Q_{m-2}(k) = (K^*)^{-1} \sum_{i=1}^{k} \Delta_i^m \{e^{(m-1)u_i} - e^{(m-1)u_{i-1}}\}/(m - 1),$$

$m = 3, 4$. These two quantities (based on the optimal u_i's) are given in Table 1 below. That these quantities appear to be converging to one with increasing k further supports our approximations.

The above results indicate that we may treat the intervals (3) and (6) (or (5) to (7)) as virtually interchangeable in large samples. On the strength of several examples in [4], there is considerable empirical evidence that the approximations are quite good even when n is small (see also, Examples 1 and 3).

6. Illustrative Examples

Example 1: To estimate homogeneity in performance, σ, of a certain electronic part subjected to continuous and constant stress, 25 such parts were subjected

Table 1
Limits of ratios of skewness and excess

k	$Q_1(k)$	$Q_2(k)$
1	1.203	1.659
2	1.119	1.376
3	1.078	1.244
4	1.055	1.172
5	1.041	1.127
6	1.032	1.098
7	1.025	1.078
8	1.021	1.064
9	1.017	1.053
10	1.015	1.045
11	1.013	1.038
12	1.011	1.033
13	1.009	1.029
14	1.008	1.025
15	1.007	1.023

to specified conditions and their failure times, in ascending order were (in hours): 0.9, 10.0, 17.9, 23.9, 24.8, 27.1, 32.9, 37.5, 49.5, 59.2, 60.4, 65.4, 69.5, 80.4, 88.3, 96.4, 134.9, 137.5, 138.2, 168.8, 172.0, 212.4, 215.0, 276.4 and 430.3. It is known from past experience that this type of part follows an exponential failure distribution, $e(x; \mu, \sigma)$ (the failure times are actually from $e(x; 0, 100)$). A 95% confidence interval is desired for σ, but in the interest of compressing the data, only the best three order statistics are to be used. Suppose (for the present) that it is known that $\mu = 0$. From Kulldorff's Table 1, with $k = 3$ and $n = 25$, we find $K = 22.9074$ and $\sigma_1^*(3, 25; 0) = -.728691 \cdot 0 + .465654 \cdot X(14) + .204237 \cdot X(22) + .058799 \cdot X(25) = 106.12$ (Notice that $\sqrt{K/n} = .9572$, REL $(3, 25; .05) = .9580$ and $\sqrt{K^*} = .9439$ are reasonably close to each other, as expected from Theorems 5 and 6). From Table 2 we get the 95% confidence interval $(.688613 \cdot 106.12, 1.570782 \cdot 106.12) = (73.08, 166.69)$ for σ. Now, if we had used the entire sample to estimate σ, we would have $\sigma_1^*(25, 25; 0) = \bar{X} = 105.18$ and the corresponding 95% confidence interval (from (4)) is $(.700 \cdot 105.18, 1.545 \cdot 105.18) = (73.6, 162.5)$. We can conclude that little information about σ was lost by using only the best three ordered observations. We will now approximate the exact interval (based on $k = 3$, $n = 25$) using the chi-square approximation of Section 5: $[2K] = 45$, so that treating $45\sigma_1^*/\sigma$ as χ^2_{45}, we obtain the approximate 95% confidence interval (from (6)) $(.688 \cdot 106.12, 1.586 \cdot 106.12) = (73.0, 168.3)$. This appears to be a quite adequate approximation.

Example 2: In the above example, suppose that the information on when the electronic parts were put on test has been lost (i.e., μ is unknown) and we still wish to use the best three observations. Theorem 1 dictates that we enter Harter's table (1961) with $k = 2$ and $n = 24$; this gives us $\sigma_1^*(3, 25; \mu_1^*) = -.683202 \cdot X(1) + .521895 \cdot X(17) + .161307 \cdot X(24) = 114.37$. Now, Theorem 2 tells us that $\sigma_1^*(3, 25; \mu_1^*)$ and $\sigma_1^*(2, 24; \mu)$ have the same distribution when optimal ranks are used. Thus, we enter Table 2 with $k = 2$ and $n = 24$ and we obtain the exact 95% ,confidence ,interval for ,σ: $(.672873 \cdot 114.37, 1.619525 \cdot 114.37) = (76.96, 185.23)$ and we see (as expected) that some precision has been lost by not knowing μ. $K = \sigma^2/\text{Var}\,\{\sigma_1^*(2, 24; \mu)\} = 20.1918$ (or $[2K] = 40$), so, if exact tables were not available, one would approximate the desired interval by treating $40\sigma_1^*(3, 25; \mu_1^*)/\sigma$ as χ^2_{40} and this yields the approximate 95% confidence interval: $(.674 \cdot 114.37, 1.637 \cdot 114.37) = (77.1, 187.2)$ and this agrees well with the above exact interval.

Example 3: Once again using the above data, we will estimate the homogeneity in performance of the electronic parts, but using the large sample theory ([10], [14] and [15]) and our chi-square approximation. With $k = 3$, $n = 25$ and $\mu = 0$, we enter table 11D.1 of [14] and find $K^* = .8910$ and $\sigma_2^*(3, 25; 0) = -.7518 \cdot 0 + .4477 \cdot X(14) + .2266 \cdot X(21) + .0775 \cdot X(25) = 108.3$. Treating $[2nK^*]\sigma_2^*/\sigma = 44\sigma_2^*/\sigma$ as χ^2_{44}, we calculate the approximate 95% confidence interval for σ: $(.685 \cdot 108.3, 1.596 \cdot 108.3) = (74.2, 172.8)$, and this is reasonably close to the exact interval of Example 1.

TABLE 2

Exact 100(1 − α)% confidence intervals for σ based on the k optimum ranks from a random sample of size n with μ known. When μ is unknown, the table applies for the k + 1 optimum ranks from a random sample of size n + 1. Entries accompanied by an asterisk (*) are approximate

k = 1

n	$c_1(.10)$	$c_2(.10)$	REL	$c_1(.05)$	$c_2(.05)$	REL	$c_1(.01)$	$c_2(.01)$	REL
1	.333808	19.49573	1.0000	.271085	39.49789	1.0000	.188739	199.4996	1.0000
2	.408037	5.926602	.9428	.342801	8.715328	.9442	.250409	20.45403	.9422
3	.449639	3.989833	.9017	.383617	5.300539	.9017	.286674	9.776458	.9022
4	.477512	3.253636	.8687	.411262	4.109993	.8689	.311741	6.739301	.8696
5	.498026	2.865222	.8413	.431779	3.510519	.8417	.330644	5.365888	.8424
6	.524031	2.677141	.8014	.461747	3.263140	.7891	.364486	4.948279	.7593
7	.543563	2.440875	.8115	.481927	2.912336	.8029	.384294	4.208661	.7816
8	.559438	2.279599	.8145	.498424	2.677920	.8082	.400641	3.737567	.7928
9	.572694	2.161954	.8138	.512261	2.509626	.8090	.414494	3.411015	.7969
10	.585929	2.103160	.7952	.527887	2.435124	.7865	.433453	3.295958	.7655
11	.598237	2 013521	.8017	.540921	2.308955	.7947	.446902	3.059585	.7776
12	.608918	1.942589	.8053	.552269	2.210162	.7995	.458659	2.878552	.7856
13	.618298	1.884920	.8067	.562269	2.130473	.8021	.469104	2.735203	.7906
14	.626634	1.836986	.8070	.571174	2.064715	.8031	.478442	2.618905	.7932
15	.635500	1.807612	.7993	.581697	2.028217	.7937	.491554	2.565150	.7800
16	.643369	1.767051	.8023	.590172	1.973050	.7974	.500596	2.469204	.7856
17	.650479	1.732242	.8041	.597850	1.925980	.7999	.508826	2.388251	.7895
18	.656946	1.702005	.8049	.604846	1.885242	.8013	.516361	2.318968	.7923
19	.662865	1.675456	.8051	.611258	1.849644	.8019	.523287	2.258904	.7940
20	.669325	1.657403	.8011	.618936	1.827436	.7969	.532939	2.227036	.7869
21	.674970	1.633831	.8027	.625088	1.795988	.7990	.539666	2.174595	.7899
22	.680184	1.612784	.8038	.630779	1.768014	.8004	.545897	2.128304	.7922
23	.685024	1.593855	.8043	.636066	1.742921	.8014	.551719	2.087031	.7938
24	.689533	1.576724	.8045	.640998	1.720277	.8018	.557153	2.050005	.7952
25	.694514	1.564300	.8020	.646922	1.705097	.7987	.564647	2.028545	.7906
26	.698842	1.548681	.8031	.651679	1.684529	.8001	.569940	1.995162	.7928
27	.702899	1.534395	.8038	.656137	1.665757	.8010	.574901	1.964874	.7942
28	.706712	1.521266	.8041	.660337	1.648560	.8016	.579590	1.937284	.7953
29	.710305	1.509169	.8042	.664296	1.632719	.8018	.584019	1.911941	.7960
30	.714301	1.499997	.8026	.669050	1.621551	.7999	.590056	1.896280	.7931
31	.717769	1.488764	.8033	.672884	1.606905	.8008	.594382	1.873008	.7945
32	.721054	1.478332	.8037	.676516	1.593321	.8014	.598467	1.851493	.7956
33	.724167	1.468615	.8040	.679964	1.590680	.8018	.602365	1.831584	.7966
34	.727127	1.459534	.8041	.683246	1.568907	.8020	.606080	1.813051	.7971
35	.730426	1.452428	.8029	.687170	1.560267	.8006	.611084	1.801028	.7950
36	733294	1.443901	.8035	.690356	1.549229	.8013	.614705	1.783751	.7960
37	.736027	1.435891	.8038	.693392	1.538870	.8017	.618165	1.767617	.7969
38	.738638	1.428355	.8040	.696299	1.529140	.8021	.621468	1.752440	.7972
39	.741387	1.422670	.8026	.699567	1.522240	.8005	.625652	1.742921	.7950
40	.743922	1.415539	.8032	.702391	1.513050	.8012	.626888	1.728639	.7962

TABLE 2 *Continued*

$k = 1$

n	$c_1(.10)$	$c_2(.10)$	REL	$c_1(.05)$	$c_2(.05)$	REL	$c_1(.01)$	$c_2(.01)$	REL
41	.746348	1.408804	.8036	.705099	1.504372	.8017	.631987	1.714888	.7972
42	.748674	1.402428	.8039	.707694	1.496183	.8021	.634977	1.703034	.7973
43	.750907	1.396378	.8040	.710187	1.488395	.8023	.637832	1.690266	.7983
44	.753274	1.391698	.8030	.713006	1.482780	.8011	.641438	1.685424	.7945
45	.755447	1.385928	.8034	.715434	1.475336	.8016	.644240	1.671766	.7969
46	.757539	1.380450	.8037	.717776	1.468310	.8020	.646945	1.660564	.7980
47	.759552	1.375229	.8039	.720028	1.461645	.8022	.649559	1.651076	.7980
48	.761492	1.370237	.8041	.722202	1.455191	.8026	.652070	1.637233	.8016
49	.763561	1.366342	.8031	.724665	1.450725	.8013	.654*	1.628*	.795*
50	.765453	1.361495	.8036	.726788	1.444173	.8022	.656*	1.620*	.796*
∞	1.000000	1.000000	.8047	1.000000	1.000000	.8047	1.000000	1.000000	.8047

$k = 2$

n	$c_1(.10)$	$c_2(.10)$	REL	$c_1(.05)$	$c_2(.05)$	REL	$c_1(.01)$	$c_2(.01)$	REL
2	.42159	5.6243	1.0000	.35896	8.2645	1.0000	.26918	19.3050	1.0000
3	.472908	3.708892	.9864	.410918	4.905481	.9864	.318176	8.991545	.9871
4	.507480	2.993964	.9699	.446224	3.760035	.9699	.352067	6.112578	.9702
5	.536464	2.600570	.9649	.476737	3.162896	.9648	.383183	4.780509	.9646
6	.559246	2.357417	.9596	.500699	2.804662	.9594	.407547	4.034661	.9596
7	.577527	2.193866	.9525	.519995	2.568503	.9525	.427310	3.565275	.9526
8	.593191	2.075932	.9449	.536811	2.401290	.9447	.445133	3.245401	.9448
9	.607527	1.979818	.9424	.552199	2.267017	.9423	.461424	2.996993	.9418
10	.619791	1.904845	.9388	.565390	2.163235	.9388	.475449	2.808997	.9390
11	.630441	1.844600	.9345	.576866	2.080481	.9345	.487700	2.661685	.9345
12	.640235	1.795375	.9297	.587603	2.013679	.9295	.499544	2.545384	.9293
13	.649382	1.750942	.9278	.597564	1.953627	.9276	.510402	2.442116	.9274
14	.657553	1.713262	.9252	.606473	1.902938	.9251	.520140	2.355864	.9248
15	.664910	1.680860	.9222	.614505	1.859533	.9222	.528942	2.282699	.9223
16	.672908	1.657031	.9161	.623862	1.829846	.9144	.540438	2.239045	.9105
17	.679843	1.629012	.9165	.631481	1.792533	.9150	.548903	2.176914	.9114
18	.686180	1.604351	.9162	.638450	1.759803	.9150	.556664	2.122838	.9119
19	.692290	1.582002	.9163	.645219	1.730348	.9152	.564300	2.074810	.9123
20	.697936	1.561702	.9164	.651462	1.703617	.9153	.571331	2.031352	.9130
21	.703161	1.543452	.9160	.657244	1.679646	.9151	.577852	1.992608	.9129
22	.708054	1.527128	.9152	.662684	1.658311	.9143	.584032	1.958476	.9121
23	.712792	1.511579	.9151	.667951	1.638011	.9144	.590025	1.926037	.9122
24	.717216	1.497388	.9148	.672873	1.619525	.9141	.595634	1.896855	.9123
25	.721361	1.484378	.9142	.677486	1.602602	.9136	.602*	1.865*	.912*
26	.725307	1.472520	.9134	.681900	1.587260	.9127	.606*	1.832*	.912*
27	.729131	1.461084	.9131	.686171	1.572460	.9125	.610884	1.823012	.9107
28	.732737	1.450481	.9126	.690202	1.558721	.9121	.615522	1.799649	.9119
29	.736*	1.439*	.913*	.694*	1.546*	.912*	.620*	1.780*	.912*
30	.740*	1.430*	.913*	.698599	1.533946	.9120	.624*	1.765*	.912*
∞	1.000000	1.000000	.9057	1.000000	1.000000	.9057	1.000000	1.000000	.9057

TABLE 2 *Continued*

$k = 3$

n	$c_1(.10)$	$c_2(.10)$	REL	$c_1(.05)$	$c_2(.05)$	REL	$c_1(.01)$	$c_2(.01)$	REL
3	.47649	3.6697	1.0000	.41525	4.8497	1.0000	.32349	8.8731	1.0000
4	.514452	2.938568	.9948	.454523	3.685041	.9947	.362240	5.977308	.9952
5	.543289	2.556025	.9895	.484703	3.103784	.9895	.392687	4.679388	.9895
6	.566895	2.315944	.9866	.509670	2.750609	.9864	.418393	3.945961	.9867
7	.585964	2.153950	.9819	.529916	2.517145	.9819	.439408	3.483456	.9820
8	.602166	2.035647	.9774	.547245	2.349492	.9773	.457665	3.163488	.9777
9	.616936	1.942426	.9757	.563250	2.219504	.9757	.474974	2.923659	.9752
10	.629910	1.867746	.9746	.577299	2.116422	.9746	.490136	2.737854	.9748
11	.641216	1.807634	.9727	.589566	2.034093	.9727	.503427	2.592059	.9727
12	.651569	1.756980	.9715	.600862	1.965315	.9715	.515808	2.472485	.9716
13	.660842	1.714243	.9702	.610996	1.907620	.9701	.526952	2.373439	.9702
14	.669155	1.677877	.9683	.620095	1.858765	.9683	.536986	2.290558	.9682
15	.676883	1.645993	.9668	.628588	1.816188	.9668	.546430	2.219190	.9670
16	.684000	1.617848	.9654	.636422	1.778749	.9654	.555171	2.156907	.9656
17	.690584	1.593391	.9635	.643730	1.746428	.9635	.563462	2.104020	.9631
18	.696827	1.570705	.9626	.650637	1.716478	.9626	.571245	2.054356	.9630
19	.702723	1.550072	.9621	.657173	1.689403	.9621	.578*	2.009*	.962*
20	.708176	1.531516	.9614	.663224	1.665043	.9613	.585505	1.971838	.9615
21	.713238	1.514770	.9603	.668845	1.643202	.9602	.592*	1.960*	.960*
22	.718101	1.499232	.9596	.673*	1.621*	.960*	.597*	1.950*	.960*
23	.721*	1.488*	.959*	.679321	1.604282	.9589	.603*	1.935*	.959*
24	.726*	1.471*	.958*	.684*	1.588*	.958*	.609*	1.870*	.958*
25	.730*	1.460*	.958*	.688613	1.570782	.9580	.614*	1 845*	.958*
∞	1.000000	1.000000	.9439	1.000000	1.000000	.9439	1.000000	1.000000	.9439

$k = 4$

4	.51589	2.9274	1.0000	.45622	3.6697	1.0000	.36438	5.9524	1.0000
5	.545536	2.542117	.9977	.487366	3.085387	.9975	.395959	4.648026	.9974
6	.569366	2.303125	.9952	.512601	2.733934	.9952	.422023	3.918669	.9950
7	.589134	2.139486	.9929	.533676	2.498559	.9929	.444086	3.453870	.9928
8	.605934	2.019959	.9909	.551703	2.329449	.9908	.463209	3.132116	.9913
9	.620561	1.927892	.9893	.567473	2.200940	.9893	.480096	2.894764	.9889
10	.633587	1.854167	.9884	.581595	2.099192	.9884	.495381	2.711432	.9888
11	.644981	1.794659	.9869	.593975	2.017722	.9869	.508845	2.567283	.9870
12	.655150	1.745269	.9852	.605057	1.950596	.9851	.520970	2.450402	.9854
13	.664441	1.703073	.9840	.615217	1.893629	.9839	.532169	2.352375	.9842
14	.672*	1.673*	.982*	.624416	1.845319	.9824	.543*	2.260*	.982*
15	.680809	1.634967	.9819	.632*	1.815*	.982*	.553*	2.180*	.982*
∞	1.000000	1.000000	.9628	1.000000	1.000000	.9628	1.000000	1.000000	.9628

TABLE 2 *Continued*

$k = 5$

n	$c_1(.10)$	$c_2(.10)$	REL	$c_1(.05)$	$c_2(.05)$	REL	$c_1(.01)$	$c_2(.01)$	REL
5	.54624	2.5381	1.0000	.48821	3.0798	1.0000	.39701	4.6382	1.0000
6	.570329	2.298191	.9986	.513745	2.727516	.9986	.423449	3.908164	.9984
7	.590361	2.133980	.9972	.535134	2.491467	.9972	.445910	3.442582	.9972
8	.607304	2.014293	.9958	.553320	2.322211	.9957	.465214	3.120757	.9958
9	.622144	1.921884	.9950	.569343	2.193313	.9950	.482421	2.883010	.9949
10	.635076	1.848925	.9938	.583354	2.277761	.9938	.497574	2.701381	.9938
11	.646633	1.789205	.9931	.595927	2.010875	.9930	.511286	2.557022	.9928
12	.656983	1.739543	.9921	.607229	1.943376	.9921	.514*	2 545*	.992*
13	.666352	1.697408	.9912	617482	1.886607	.9911	.527*	2.428*	.991*
14	.669*	1.691*	.990*	.620*	1.878*	.990*	.538*	2.330*	.990*
15	.677*	1.654*	.980*	.630*	1.829*	.980*	.549*	2.247*	.980*
∞	1.000000	1.000000	.9734	1.000000	1.000000	.9734	1.000000	1.000000	.9734

7. CONCLUSIONS

We conclude that the high efficiency of point estimates of σ based on optimally selected order statistics carries over into interval estimation of σ based on these estimates. We have seen that, in large samples, the same quantiles are optimal for both point and interval estimation of σ and that various optimality criteria for selecting these quantiles are equivalent.

Finally, it is worth pointing out that the chi-square approximation of Section 5 is surely most valuable when Table 2 is inapplicable because the sample size is too large or other than the optimal ranks are available.

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