Probability of Equipment Failure When Tolerances are Incorrect

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This report describes an investigation of how errors in components of an assembly can affect its performance. In particular the report deals with the situation, uncommon in engineering practice, where the output tolerance of the assembly may be violated even though the tolerances on the components are all met. This situation is analyzed to estimate the probability that the output tolerance will be satisfied given that the component tolerances are met. Three methods are described for estimating this probability, their results are compared in a number of cases, and a best method is chosen. Several simple rules, suitable for preliminary estimates, are also given.

KEY WORDS
Design
Tolerances
Probability
Estimation
Moments
Characteristic functions
Monte Carlo Simulation

1. INTRODUCTION

This report deals with certain aspects of the general problem of errors and tolerances in the design and testing of equipment. It is presumed that the piece of equipment is required to operate at a certain level of output. Ordinarily the designer assigns a certain error-tolerance to this output, chosen so that the equipment will function properly if the output error satisfies its tolerance. The output error usually arises from errors in the various components that have been assembled to make the piece of equipment. The designer will customarily know the relation between the output error and the component errors. Common practice (see Bowker and Lieberman [1]) is that the designer will combine this relation with the output tolerance to find tolerances on each component such that satisfaction of these component tolerances will ensure that the output tolerance is met.

We are concerned here with the uncommon situation where satisfaction of the component tolerances does not ensure satisfaction of the output tolerance. This state of affairs can arise when an error has been made in choosing the component tolerances, or when it is impractical (or too expensive) to make the component tolerances small enough. In either case we must face the possibility that all the components will meet their tolerances, but some of the assembled pieces of equipment will not work properly. The practical information that we want is the probability that the output tolerance will be satisfied. With this information we can

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estimate how many extra pieces of equipment must be manufactured on the average in order to obtain a given number of workable assemblies.

In the following section we shall describe the general procedure for estimating the probability that the output satisfies its tolerance, supposing that each component error is normally distributed with zero mean, known variance and known tolerance. Three mathematical methods are given for carrying out the calculations. One is of Monte-Carlo type and is described in Section 3. The other two methods use the Characteristic Function in different ways. Section 4 gives formulas for the Characteristic Functions of the various distributions, and Sections 5 and 6 use these formulas in estimating the desired probability. Various simple approximations and limiting cases are examined in Section 7. Section 8 describes the computations, whose results are then discussed in Section 9.

2. ANALYSIS OF THE PROBLEM

We let $Y_j$ be the error in the $j$-th component, $j = 1, 2, \cdots N$, and $X_0$ be the error in the output. The relation between the output error and the component errors is taken as linear,

\[ X_0 = \sum_{j=1}^{N} C_i Y_j \]  

(1)

where the $C_i$ are assumed to be known constants.

It is assumed initially that $Y_j$ is normally distributed with zero mean and variance $\sigma_j^2$. Then we may define

\[ X_i = C_i Y_j \]

\[ S_i = |C_i| \sigma_i \]

and the relation (1) can be written

\[ X_0 = \sum_{i=1}^{N} X_i \]  

(3)

where $X_i$ is normally distributed with zero mean and variance $S_i^2$.

We let $D_i > 0$ be the tolerance on the error, $Y_j$, in the $j$-th component, and $B_0 > 0$ be the tolerance on the output error, $X_0$. Thus, when $Y_j$ satisfies its tolerance, we have

\[ |Y_j| \leq D_i \]

and, if $X_0$ satisfies its tolerance, then

\[ |X_0| \leq B_0 \]

We define also

\[ B_i = |C_i| D_i \]  

(4)

as the tolerance on $X_i$, so that, if $X_i$ satisfies its tolerance, then

\[ |X_i| < B_i \]

We notice also that

\[ B_i / S_i = D_i / \sigma_i \]  

(5)

We now define certain probabilities
PROBABILITY OF EQUIPMENT FAILURE WHEN TOLERANCES ARE INCORRECT

\[ P_i = \text{prob. } |Y_j| \leq D_j = \text{prob. } |X_i| \leq B_i \]  
\[ P_e = \text{prob. } |X_i| \leq B_i \text{ for all } j \]  
\[ P_A = \text{prob. } |X_0| \leq B_0 \text{ and } |X_i| \leq B_i \text{ for all } j \]  
\[ P* = \text{prob. } |X_0| \leq B_0, \text{ given that } |X_i| \leq B_i \text{ for all } j \]

We are mainly concerned with estimating \( P* \). We assume that the \( X_i \) are independent, hence

\[ P_e = \prod_{i=1}^{N} P_i \]

The theorem on compound probability asserts that

\[ P_A = \text{prob. } |X_0| \leq B_0, \text{ given that } |X_i| \leq B_i \text{ for all } j \]

\[ \times \text{prob. } |X_i| \leq B_i \text{ for all } j \]

Then we can write (10) with the aid of (7) and (9) as

\[ P* = P_A/P_e = P_A \prod_{i=1}^{N} P_i \]

Finally, it is useful to define

\[ \Delta = \sum_{i=1}^{N} B_i = \sum_{i=1}^{N} |C_i| D_i \]

If all the components satisfy their tolerances, then we have

\[ |X_0| = \left| \sum_{i=1}^{N} X_i \right| \leq \sum_{i=1}^{N} |X_i| \leq \sum_{i=1}^{N} B_i \]

and so, because of (12), \( X_0 \) must satisfy the inequality

\[ |X_0| \leq \Delta \]

If \( \Delta \leq B_0 \) then (13) implies

\[ |X_0| \leq \Delta \leq B_0 \]

In this case we see that, if each component satisfies its tolerance, the output error \( X_0 \) must always satisfy its tolerance, and from (9) and (11) we conclude that

\[ P* = 1, \quad P_A = P_e \]

This case is the common one in design practice, i.e., the tolerances are set so that, if each component meets its tolerance, the output will necessarily satisfy its tolerance. However, in this paper we are interested in the opposite case, where

\[ \Delta > B_0 \]

and

\[ 0 \leq P* < 1 \]

Our main objective is to estimate \( P* \). We define \( f^*(x_0) \) as the density function of the output when the separate component errors all satisfy their tolerances. Since the component errors are normally distributed, their density functions, when they satisfy their tolerances, are symmetrically-truncated normal distributions, and \( f^*(x_0) \) is the density function of a finite sum of these quantities. \( P* \) is the integral
between \(-B_0\) and \(B_0\) of \(f^*(x_0)\). The main difficulty in estimating \(P^*\) is that \(f^*(x_0)\) is not easily expressible as a function of the parameters of the component density functions.

3. **Monte Carlo Method**

This procedure for estimating \(P^*\) was carried out by a digital computer program. Random numbers, normally distributed with zero means and the desired variances, were generated and taken as the values of \(Y_j, j = 1, \ldots, N\). Each \(Y_i\) was tested to see if it satisfied \(|Y_i| \leq D_i\), then \(X_0\) was calculated from (1) and tested to see if it satisfied \(X_0 \leq B_0\). The entire process was repeated \(L\) times with different sets of values for the \(Y_i\) and counts were made of \(n, \) the number of samples for which \(|Y_j| \leq D_j\) for all \(j = 1, \ldots, N,\) and \(\eta\), the number of samples where \(|X_0| \leq B_0\). Then \(P^*\) was estimated from \(P^* = \eta/n\).

4. **Characteristic Functions**

The remaining two methods of estimating \(P^*\) use the Characteristic Function (Fourier Transform) as the main tool in the analysis. In this section we summarize certain relationships that form the basis of these methods.

If \(f_i(X_i)\) is the truncated density function for the variable \(X_i\), then its Characteristic Function (C.F.) is defined by

\[
\phi_i(t) = \int_{-\infty}^{\infty} e^{-itx}f_i(x)\, dx
\]

Similarly \(\phi^*(t)\) is the C.F. of \(f^*(X_0)\). Because the \(X_i\) are independent, we have

\[
\phi^*(t) = \prod_{j=1}^{N} \phi_j(t)
\]

Also, from the Complex Inversion Relation for Fourier Transforms we may derive

\[
P^* = \int_{-B_0}^{B_0} f^*(x_0)\, dx_0 = \pi^{-1} \int_{-\infty}^{\infty} \phi^*(t) t^{-1} \sin (tB_0)\, dt
\]

A series representation of \(\phi_i(t)\) may be derived by expanding its complex, Error Function, representation in a Taylor Series,

\[
\phi_i(t) = e^{-\gamma_i} \left\{ 1 - \frac{2e^{-\gamma_i}}{\pi^2 \text{erf} \gamma_i} \sum_{n=1}^{\infty} (-\rho_i)^n \frac{H_{2n}(-\gamma_i)}{(2n)!} \right\}
\]

where \(\gamma_j = B_j/(2^jS_j)\) and \(\rho_j = tS_j/2^j\) and \(H_k\) is the Hermite Polynomial of degree \(k\). Expanding \(e^{-\gamma_i^2}\), we obtain also the leading terms (up to \(t^2\)) in the expansions of \(\phi_i(t)\) about \(t = 0,\)

\[
\phi_i(t) = 1 - S^2 \left( 1 - \frac{2\gamma_i e^{-\gamma_i^2}}{\pi^2 \text{erf} \gamma_i} \right) (t^2/2!) + S^3 \left( 3 - \frac{2\gamma_i(3 + \gamma_i^2)e^{-\gamma_i^2}}{\pi^2 \text{erf} \gamma_i} \right) (t^4/4!) - \cdots
\]

The series in (17) converges absolutely for any finite values of \(\gamma_i\) and \(\rho_j,\) but the convergence is slow when \(\rho_j\) is large. In this case we use the complex \(W\)-function, which is defined by (see [2], for example)

\[
W(z) = W_r(z) + iW_i(z) = e^{-z} \text{erfc} (-iz)
\]

We can express \(\phi_i(t)\) as
\( \phi_i(t) = \frac{1}{\sqrt{\pi} \gamma_i} \left[ e^{-\frac{t^2}{\gamma_i}} - e^{-\gamma_i t} \right] \left[ W_r(\rho_i + i\gamma_i) \cos(B_i t) - W_i(\rho_i + i\gamma_i) \sin(B_i t) \right] \)  

(20)

A rational approximation for \( W \), accurate when \(|\rho_i + i\gamma_i| > 4\) is given in [2]; we obtain from it

\[
W_r(\rho_i + i\gamma_i) = \sum_{k=1}^{3} \gamma_k \left\{ (-\gamma_i \alpha_{ki} + \rho_i B_i)/\left( \alpha_{ki}^2 + B_i^2 \right) \right\}
\]

\[
W_i(\rho_i + i\gamma_i) = \sum_{k=1}^{3} \gamma_k \left\{ (\rho_i \alpha_{ki} + \rho_i B_i)/\left( \alpha_{ki}^2 + B_i^2 \right) \right\}
\]

(21)

\[\alpha_{ki} = \rho_i^2 - \gamma_i^2 \eta_k\]

where \( r_1 = 0.4613135, r_2 = 0.09999216, r_3 = 0.002883894, \eta_1 = -1, \eta_2 = 1.7844927 \) and \( \eta_3 = 5.5253437 \).

5. APPROXIMATION USING MOMENTS

This method consists of finding the moments \( M_1, M_2, M_3, M_4 \) of the exact distribution, \( f^*(x) \), then constructing an approximate distribution, \( g(x) \), which has the same low-order moments, and from which we can find \( P^* \) conveniently. The approximate distribution was taken as

\[
g(x) = S^{-1}(2\pi)^{-1/2} \left[ G_0 + G_2(x/S)^2 + G_4(x/S)^4 \right] e^{-x^2/(2S^2)}
\]

(22)

where \( G_0, G_2 \), and \( G_4 \) are constants to be evaluated and \( S^2 \) is the exact second moment of the output. This form of distribution was chosen because all its moments exist and can be expressed as linear combinations of the \( G \)'s, and because it embraces (when \( G_2 \to G_4 \to 0 \)) the normal form to which this distribution tends in several limiting cases.

The exact moments are determined by using (15) in

\[ M_k = (-i)^k d^k \phi^*(0)/d\theta^k \]  

whence \( M_2 = 1, M_1 = M_3 = 0 \),

\[ M_2 = S^2 = -\phi^{*''}(0) = \sum_{i=1}^{N} [-\phi^{*''}(0)] \]

(24)

\[ M_4 = \phi^{*IV}(0) = 3[\phi^{*'''}(0)]^2 + \sum_{i=1}^{N} [\phi^{*V}(0) - 3(\phi^{*''}(0))^2] \]

(25)

From (18) we find

\[ -\phi^{*''}(0) = S_i^4 \left[ 1 - \left( z_i A'(z_i)/A(z_i) \right) \right] \]

(26)

\[ \phi^{*IV}(0) = S_i^6 \left[ 1 - \left( z_i A'(z_i)/A(z_i) \right)(3 + z_i^2) \right] \]

(27)

\[ z_i = B_i/S_i = 2\gamma_i \]

(28)

\[ A(z_i) = (2\pi)^{-1} \int_{-\infty}^{\infty} \lambda e^{-\lambda z_i} d\lambda; \quad A'(z_i) = (2\pi)^{1/2} \]

(29)

and the exact moments are found by inserting these in (24) and (25).

The conditions that the moments derived from \( g(x) \) are equal to the exact ones lead to three linear algebraic equations in the three unknowns \( G_0, G_2, \) and \( G_4 \).

We may solve these, substitute back into (22), and then evaluate \( P^* \), using \( g(x) \),
to get
\[ P^*(z) \approx A(z) - Hz(3 - z^2)A'(z) \quad (30) \]
\[ H = \frac{3 - (M_4/M_2^2)}{24}, \quad z = B_0/S \quad (31) \]
We shall call the estimate of \( P^* \) given by (30) the moment approximation.

6. Numerical Integration of the Characteristic Function

This procedure consists merely of carrying out the integration of (15), i.e., evaluating
\[ \varphi(t) = \frac{(2/\pi) \int_{0}^{\infty} \phi^*(t) \sin(B_0t) dt}{\varphi(t)} \quad (32) \]
where \( \phi^*(t) \) is calculated from the \( \phi_j(t) \) by (15). The \( \phi_j(t) \) are evaluated by use of (17) when \( \rho_i \leq 4 \) and (20) and (21) when \( \rho_i > 4 \).

In general it is necessary to evaluate the integral by numerical means. This is occasionally troublesome, not only because \( \sin(B_0t) \) is oscillatory but also because \( \phi^*(t) \) is of damped, oscillatory type. In qualitative terms, if some of the component tolerances are large, then \( \phi^*(t) \) becomes and remains small from some \( t \) onward, and the principal contribution to the integral comes from a region fairly near \( t = 0 \). The numerical integration is not difficult in this case. However, if all the component tolerances are small, then \( \phi^*(t) \) dies away very slowly and non-negligible contributions may come from a broad range of \( t \). In this case the oscillatory behavior of the integrand makes the numerical integration somewhat exacting.

The following numerical procedure was adopted. The range of integration was divided into intervals of length \( T \), and in each interval the integral was evaluated by using Gaussian Integration with a sufficiently large number of points (typically 20). The basic length \( T \) was chosen small enough so that for each case the integrand oscillated only a few times (i.e. had four or fewer zeroes) in any fundamental interval. The number of intervals was chosen large enough so that the union of the intervals covered the entire range in which the integrand made a significant contribution to the integral. A good deal of experimentation was done, using different sizes and numbers of fundamental intervals and various numbers of points in each interval. In all cases error bounds for \( P^* \) could be estimated quite accurately. The error was never greater than \( 5 \times 10^{-5} \) and usually much less than that; this is much better accuracy than we can expect with the other methods and permits us to regard the results of this method as the “correct” ones.

7. Simple Approximations and Limiting Cases

A number of simple approximations for \( P^* \) can be derived for certain special situations, and we describe them briefly here.

(i) If \( N \) is very large, and \( X_a \) is not dominated by a few components, the Central Limit Theorem leads us to expect that \( \phi^*(x) \) will be of normal form.

(ii) If all the component tolerances are very large, i.e. if
\[ \frac{D_i}{\sigma_i} = \frac{R_i}{S_i} \gg 1 \]
then \( \phi^*(x) \) is approximately a normal distribution with zero mean and variance \( S_i \), where
\[ S_i^2 = \sum_{j=1}^{N} S_j^2. \quad (33) \]
This leads to the following estimate of $P^*$:

$$P^*_t(B_0/S_t) = A(B_0/S_t)$$

(iii) If one component, say $X_k$, dominates the rest, i.e. $S_k \gg S_j$ for all $j \neq k$, the distribution of $X_k$ is approximately that of $X_k$ truncated at $|X_k| = B_k$. This gives the following estimate of $P^*$:

$$P^*_t(B_0/S_t) = A(B_0/S_t)/A(B_k/S_k) \quad B_0 < B_k$$

$$= 1 \quad B_0 \geq B_k$$

(iv) Another simple approximation is obtained by setting $G_s = 0$ in (22) and choosing $G_s$ and $G_2$ such that

$$\int_{-\infty}^{\infty} x^{2k} g(x) \, dx = M_{2k} \quad k = 0, 1$$

Then we get as an approximation for $P^*$ merely the first term of (30),

$$P^*_t(B_0/S) = A(B_0/S)$$

where $S = M_1^2$. This amounts to deriving the estimate of $P^*$ by first finding the exact variance of $X_0$ and then assuming that $X_0$ is normally distributed with that variance.

8. Computations and Results

The results of problems of this type are fairly easily estimated when the number of components is large and again when it is small (i.e. $N = 1$ or 2). The intermediate region is the most awkward, and for that reason all cases discussed here have $N = 4$, i.e. lie in that region. For simplicity $\sigma_j = 1$ for all $j$ in all cases. Eleven different cases were studied with values for $C_j$ and $D_j$ as shown in Table I.

The results are shown in Figures 1 to 4 as $P^*$ plotted versus $B_0/S_t$. Each graph shows the moment approximations, (30), as curves and the Monte-Carlo results (for $L = 2000$ samples) as plotted points. The parameter $S_t$, i.e. the standard deviation of $X_0$ if none of the components had been truncated, was used as reference here in order to give a convenient basis for comparison among the various cases.

The method of integrating the Characteristic Function was used to spotcheck the other results, at points where the moment approximation and Monte-Carlo procedure did not agree well. The comparison of the three is shown in Table II.

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<th>Case</th>
<th>$C_1$</th>
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<th>$C_3$</th>
<th>$C_4$</th>
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Comparison among Values of $P^*$ given by the three Methods of Calculation for various Cases and Tolerances.

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9. DISCUSSION

We shall first compare the results obtained by the three methods, then suggest procedures for estimating $P^*$ under various conditions.

Figures 1 to 4 show that the agreement between the Monte-Carlo method and moment approximation is reasonably good most of the time. We see from Table 2 that, when these results do not agree well, the integration of the characteristic function usually agrees better with the moment approximation than the Monte-Carlo Method.

The results suggest that with the Monte-Carlo Method $L = 2000$ samples were not sufficient to give two-place accuracy. A larger sample would have been desirable but was quite impractical on the computing equipment that was available.

In general the moment approximation gives decent estimates of $P^*$. However, it has two (related) theoretical defects that are worth mentioning. First the approximate density function, $f^*(x_o)$ is continuous (see Equation (22)), but the true density function, $f^*(x_o)$, is discontinuous and in fact $f^*(x_o) = 0$ for $x_o > \Delta$. Second,
FIGURE 2—$P^*(B_0/S_1)$ in Cases IV and V. Open and Solid Circles are Monte Carlo estimates in Case IV and V respectively. The moment approximations in Cases IV and V are coincident.

FIGURE 3—$P^*(B_0/S_1)$ in Cases VI, VII and VIII.
in many cases $g(X_n)$ is slightly negative for certain, sufficiently large, values of $x_0$. These defects are likely to have their most serious effect when the truncation points of $f^*(x_0)$ are closest to zero, i.e. when $\Delta$ is not too large. Among the cases studied this situation is realized when there is a single, dominant component with a low tolerance on it, as in Case VIII. We see from Table 2 that indeed the biggest discrepancy between the moment-approximation and ("correct") characteristic function estimates occurs for Case VIII, $B_0/S_1 = .945$. At this point the absolute difference is .012, whereas the worst error in the other cases of Table 2 is only .004.

Figure 1 shows how curves for $P^*$ change as the common tolerance value for the four components increases from $B_0/S_1 = 1$ through 1.5 to 2. As we expect, the curves become lower and tend toward the normal curve, given by $P^*$, with increasing component tolerances.

The effect on $P^*$ of an increasingly dominant component is displayed in Figures 3 and 4. Figure 3 shows the case where the increasingly dominant component has a smaller tolerance than the other components. As $C_1$ increases from 1 through 2 to 5, the curve of $P^*$ is raised toward the curve for $P^*_1$, truncated at $B_0/S_1 = 1$, to which it must ultimately tend. In contrast Figure 4 shows what happens when the increasingly dominant component has a higher tolerance than the others. As $C_1$ increases from 1 through 2 to 5, the curve for $P^*$ is lowered toward the curve for $P^*_2$ truncated at $B_0/S_1 = 2$, to which it ultimately tends.

When all the $C_i$ are roughly the same, we may also inquire about the effect of changing the component tolerances but keeping the average component tolerance constant. Comparing Cases (II), (IV) and (V) in Figures 1 and 2, we see that this has scarcely any effect on $P^*$. In other words, when the $C_i$ are roughly equal, the mean component tolerance has a considerable influence on $P^*$ (see Figure 1) but the variance in the component tolerances has negligible effect.
A reasonably extensive comparison of the simple approximations, (33), (34) and (35) with the more accurate calculations suggests the following as a rule.

(i) Use $P_1^*$ if one component dominates greatly.
(ii) Use $P_j^*$ if $B_i/S_i \geq 2$ for all $j$.
(iii) Use $P_{11}^*$ if no one component dominates, and some $B_i/S_i < 2$.

Use of this rule will give fair results, perhaps suitable for an initial estimate. The simplest, tolerable accurate procedure is the moment-approximation, given by (30) and (31).

References
