Comparison of Approximate Confidence Intervals for the Exponential Scale Parameter from Sample Quantiles

KENNETH S. KAMINSKY
Bucknell University
Lewisburg, Pennsylvania

Several authors have considered point and interval estimation of the exponential scale parameter, $\sigma$, on the basis of subsets of the order statistics. In this paper, we suggest a procedure for finding approximate confidence intervals for $\sigma$ in large samples, using $k$ sample quantiles of a random sample of size $n$. The procedure is based on a simple chi-square approximation to the distribution of the asymptotically best linear estimate of $\sigma$. We compare this procedure with the one given by Ogawa (1962) based on an approximating $t$-distribution. We find that the interval based on the chi-square approximation is easier to calculate, and performs better when $k$ is small and $n$ is large.

**KEY WORDS**
Exponential Scale Parameter
Confidence Intervals
Sample Quantiles
Order Statistics

1. INTRODUCTION

Let $[\lambda_1, \cdots, \lambda_k]$ be such that $0 < \lambda_1 < \cdots < \lambda_k < 1$, $k \geq 1$ and define $\lambda_0 = 0$ and $\lambda_{k+1} = 1$. The set of $\lambda$'s is a spacing.

If $[X(0)]_{k+1}$ are the order statistics of a random sample of size $n$ from an exponential population with p.d.f. $e(x; \sigma) = (1/\sigma) \exp (-x/\sigma)$, $x, \sigma \geq 0$, then

$$X(n_0) < X(n_2) < \cdots < X(n_k), \quad n_i = \lceil n\lambda_i \rceil + 1$$

are sample quantiles of this random sample (where $\lceil n\lambda \rceil$ is the greatest integer $\leq n\lambda$). The population $\lambda_i$-quantile of $e(x; 1)$ is $u_i = \log (1 - \lambda_i)^{-1}$, ($i = 0, 1, \cdots, k$) and $u_{k+1} = \infty$.

Ogawa (§11E of [5]) derives a statistic based on $\sigma^*$, the asymptotically best linear estimate (ABLE) of $\sigma$:

$$t = \sqrt{(k-1)K_2} \frac{(\sigma^* - \sigma)}{\sqrt{S}},$$

which is asymptotically ($n \to \infty$) a $t$-variate with $k - 1$ degrees of freedom, where

$$\sigma^* = \sum_{i=1}^k B_i X(n_i),$$

$$B_i = (\Delta_i - \Delta_{i+1})/K_2 \quad (i = 1, 2, \cdots, k),$$

$$\Delta_i = (u_i - u_{i-1})/\{\exp (u_i) - \exp (u_{i-1})\}, \quad \Delta_{k+1} = 0,$$

Received Feb. 1972; revised Sept. 1972

483
\[ K_2 = \sum_{i=1}^{k} \frac{(u_i - u_{i-1})^2}{\exp(u_i) - \exp(u_{i-1})} \]

and

\[ S = \sum_{i=1}^{k+1} \{X(u_i) \exp(-u_i) - X(u_{i-1}) \exp(-u_{i-1})\} \exp(-u_{i-1}) \exp(-u_i) \quad K_2 \sigma^2. \tag{1} \]

An approximate 100(1 - \( \alpha \))% confidence interval for \( \sigma \) based on this statistic is [5]:

\[ \sigma^* \pm t_{1-\alpha/2, k-1} \sqrt{S/\{k(1)K_2\}}, \tag{2} \]

where our convention is (as with \( z \) and \( \chi^2 \) later): \( t_i \) is the upper 100\( \% \) point of the \( t \)-distribution.

In a recent paper, Kaminsky [3] gives exact and approximate confidence intervals for \( \sigma \) based on the best linear unbiased estimate (BLUE) and on the ABLE of \( \sigma \) using optimally selected sample quantiles. The optimality criterion, in large samples, is that of selecting the set of \( \lambda \)'s which maximize \( K_2 \), with \( k \) fixed. (This amounts to minimizing the asymptotic variance of \( \sigma^* \) since \( AV(\sigma^*) = \sigma^2/(nK_2) \).)

The set of \( k \lambda \)'s (\( k = 1(1)13 \)) which maximize \( K_2 \) were found numerically by Ogawa (see §11E of [5]) together with the corresponding values of the \( B \)'s, \( u \)'s and \( K_2 \).

In large samples the approximate 100(1 - \( \alpha \))% confidence interval for \( \sigma \) is obtained by treating \( [2nK_2]\sigma^*/\sigma \) as a \( \chi^2 \) variate with \( [2nK_2] \) degrees of freedom. This amounts to matching the asymptotic values of the first two moments of \( [2nK_2]\sigma^*/\sigma \) and \( \chi^2_{[2nK_2]} \). Justification for this is discussed in some detail in [2] and [3]. To see that the moments are indeed matched when \( n \) is large we note that [2], [3]

\[ E(\sigma^*) = \left( \sum_{i=1}^{k} \Delta_i \delta_i/K_2 \right), \]

\[ \text{Var}(\sigma^*) = \left( \sum_{i=1}^{k} \Delta_i^2 \delta_i/K_2^2 \right)\sigma^2 \]

and

\[ \delta_i = (u_i - u_{i-1}) + O(n^{-1}); \quad n\delta_i = \{\exp(u_i) - \exp(u_{i-1})\} + O(n^{-1}) \quad (i = 1, 2, \cdots, k) \tag{3} \]

where

\[ \delta_i = \sum_{j=n-i+1}^{n} (n - j + 1)^{-r}, \quad r = 1, 2. \]

Therefore \( E(\sigma^*) \rightarrow \sigma \) and \( n \cdot \text{Var}(\sigma^*) \rightarrow \sigma^2/K_2 \) as \( n \rightarrow \infty \). The approximate 100(1 - \( \alpha \))% confidence interval we obtain is

\[ ([2nK_2]\sigma^*/\chi^2_{1-\alpha/2, 2nK_2} + [2nK_2]^{\sigma^2}/\chi^2_{\alpha/2, 2nK_2}). \tag{4} \]

The interval (4) has been shown to be virtually interchangeable in large samples, with the corresponding exact 100(1 - \( \alpha \))% confidence interval obtained from the exact distribution of \( \sigma^* \) [2], [3].

In this paper we compare the intervals (2) and (4) and we find that
(a) the interval (4) is considerably easier to compute than the interval (2),
(b) the interval (4) can be used for all $k$ while it is clear that (2) does not apply when $k = 1$, 
(c) the interval (4) performs better than (2) in large samples when the two are compared on the basis of the ratio of expected squared lengths. The limit of this ratio (with (4) in the numerator) is seen to be \((z_{1-a/2}/t_{1-a/2,k-1})^2\).

In section 3 we will briefly discuss the dual problem of hypothesis testing.

2. Comparison of the Intervals

Denote the expected value of the square of the lengths of the intervals (2) and (4) by \(E SL(k, n; \alpha)\) and \(E SL^*(k, n; \alpha)\) respectively. (The \(\lambda's\) are fixed throughout).

From (3) and the expressions for the mean and variance of \(\sigma^*\) we have \(E(\sigma^*) \rightarrow \sigma^2\) as \(n \rightarrow \infty\). Together with the asymptotic normality of the chi-square statistic, this implies that

\[
\lim_{n \to \infty} \frac{\sigma^2}{\lambda} = \frac{\zeta_1^2}{\lambda}.
\]

Now, \(E SL(k, n; \alpha) = 4t_{1-a/2,k-1}E(S)/(k - 1)K_2\) and it can be shown that [2]

\[
E(S) = \left(\sum_{i=1}^{k} (\delta_i + \delta_i)/\{\exp (u_i) - \exp (u_{i-1})}\right) - (1/K_2)\left(\sum_{i=1}^{k} \Delta_i \delta_i + \left(\sum_{i=1}^{k} \Delta_i \delta_i\right)^2\right)\sigma^2.
\]

Using (3) we have

\[
\lim_{n \to \infty} \frac{\sigma^2}{\lambda} = 4t_{1-a/2,k-1}/K_2.
\]

The limit of the ratio of the expected squared lengths is thus \((z_{1-a/2}/t_{1-a/2,k-1})^2\).

This is undefined when \(k = 1\) (as expected, since (2) is then undefined) and is considerably smaller than one when \(k\) is small.

Example 1: This example is based on data from Maguire, Pearson, and Wynn [4] (and reproduced on p. 378 of [5]). The data are time intervals in days between explosions in mines, involving more than 10 men killed from December 6, 1875 to May 29, 1951. Ogawa (in [5]) uses these data and chooses \(k = 10 (n = 109)\). He calculates \(\sigma^* = 242.1849 (\text{days})\) for the ABLE of \(\sigma\) based on the 10 optimum quantiles. The information needed for our calculations is presented in Table 1.

To make our calculations, we use Ogawa's Table 11D.1 [5] and a set of chi-square tables. We see that \(K_2 = .9832\) so that \(2nK_2 = 214\). The approximate 95% confidence interval from (4) is:

\[
\]

To compare the labor involved in finding the intervals (2) and (4), the reader is referred to Table 11E.1 of [5]. (Incidentally, calculating (2) is made simpler by using the fact that the sum in \(S\) (1) can be simplified to [2]

\[
\sum_{i=1}^{k} |X(n_i) - X(n_{i-1})|^2/\{\exp (u_i) - \exp (u_{i-1})\}.
\]

The approximate 95% confidence interval obtained from (2) is

\[
(154.8299, 329.5399).
\]

It is considerably longer than the interval obtained from (4). From Table 1, we find
K. S. KAMINSKY

\[ ESL(10,109; .05) = .174\sigma^2. \]

Using \( E(\sigma^4) = \frac{1}{5}(\sum_{i=1}^5 \Delta_i^2 \delta_{i1} + (\sum_{i=1}^5 \Delta_i \delta_{i1})^2)\sigma^2/K_i^2 \)
we calculate \( ESL^*(10,109; .05) = .147\sigma^2 \), so that the ratio of the expected
lengths is .84. This is reasonably close to the limiting value of \((1.96/2.2622)^2 = .75.\)

3. APPLICATION TO HYPOTHESIS TESTING

Clearly, the approximations mentioned above can be applied to testing hy-
potheses of the form \( H_0 : \sigma = \sigma_0 \) versus scale alternatives. One advantage of the
chi-square approximation is that the power function for the test is approximated
from the chi-square distribution, while the power function for the test using the
t-statistic is approximated by the noncentral t-distribution. For example, to test
\( H_0 : \sigma = \sigma_0 \) against the alternative \( H_1 : \sigma > \sigma_0 \) using the chi-square approximation,
one rejects \( H_0 \) at the 100\(a\)% level of significance if and only if \( [2nK_2]\sigma^*/\sigma_0 > x_{1-a/2,109}\). The power function for this test is approximated in large samples by
\( \Pi(\sigma) = P(X^2_{2nK}\sigma^*/\sigma_0 > x_{1-\alpha/2,109}) \). This is significantly easier to deal with than
tables of the noncentral t-distribution.

Example 2: We return to the mine explosion data of the last example to test
the hypothesis that \( \sigma = 240 \) against the alternative hypothesis that \( \sigma \neq 240 \)
with level of significance \( \alpha = .05. \) The approximate test using the chi-square
approximation gives the critical region \( |\sigma^*| > 287.57 \) or \( \sigma^* < 196.87 \). The power function for this test is approximated by \( \Pi(\sigma) = 1 - P(X^2_{14}/214 < 287.57/\sigma, \sigma > 0) \). This function is graphed in Figure 1 (dashed curve). The test on
Ogawa's t-statistic is: reject \( H_0 \) if and only if
\[ |t| = \frac{8.8488}{\sigma^* - 240} > 2.2622. \]
When \( \sigma = \sigma_1 \neq 240 \), \( t \) has the non-central t-distribution with \( 10 - 1 = 9 \)
degrees of freedom and noncentrality parameter \( \delta = \sqrt{n}K_2(1 - \sigma_0/\sigma_1) \). With the aid of Table A-12b of [1], we have graphed the approximate power
function for this large sample t-test. (See Figure 1, solid curve).

### Table 1

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i(n_i) )</th>
<th>( \delta_{i1} )</th>
<th>( \delta_{21} )</th>
<th>( u_i )</th>
<th>( \delta_i )</th>
<th>( \delta_{i1}^2 )</th>
<th>( \delta_{21}^2 )</th>
<th>( u_i^2 )</th>
<th>( u_i^2 - u_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>1</td>
<td>26</td>
<td>50</td>
<td>0.27108</td>
<td>0.00228</td>
<td>0.2719</td>
<td>0.87020</td>
<td>0.3356</td>
<td>0.00215</td>
<td>0.34227</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>60</td>
<td>0.10580</td>
<td>0.00438</td>
<td>0.5707</td>
<td>0.65376</td>
<td>0.19992</td>
<td>0.00435</td>
<td>0.21998</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>80</td>
<td>0.23351</td>
<td>0.00609</td>
<td>0.9030</td>
<td>0.72642</td>
<td>0.15404</td>
<td>0.00601</td>
<td>0.15408</td>
</tr>
<tr>
<td>4</td>
<td>78</td>
<td>90</td>
<td>0.37774</td>
<td>0.01021</td>
<td>1.2771</td>
<td>0.32245</td>
<td>0.12565</td>
<td>0.01015</td>
<td>0.13670</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>50</td>
<td>0.41725</td>
<td>0.01849</td>
<td>1.7043</td>
<td>0.22349</td>
<td>0.09596</td>
<td>0.01839</td>
<td>0.11426</td>
</tr>
<tr>
<td>6</td>
<td>97</td>
<td>45</td>
<td>0.44453</td>
<td>0.02869</td>
<td>2.2030</td>
<td>0.14022</td>
<td>0.06133</td>
<td>0.02856</td>
<td>0.06634</td>
</tr>
<tr>
<td>7</td>
<td>103</td>
<td>74</td>
<td>0.65321</td>
<td>0.07399</td>
<td>2.8042</td>
<td>0.08054</td>
<td>0.05261</td>
<td>0.00043</td>
<td>0.06712</td>
</tr>
<tr>
<td>8</td>
<td>132</td>
<td>132</td>
<td>0.61667</td>
<td>0.13028</td>
<td>3.5581</td>
<td>0.04057</td>
<td>0.02590</td>
<td>0.00022</td>
<td>0.02747</td>
</tr>
<tr>
<td>9</td>
<td>108</td>
<td>161</td>
<td>0.83333</td>
<td>0.36111</td>
<td>4.5700</td>
<td>0.03614</td>
<td>0.01368</td>
<td>0.00010</td>
<td>0.01702</td>
</tr>
<tr>
<td>10</td>
<td>109</td>
<td>160</td>
<td>1.00000</td>
<td>1.00000</td>
<td>6.1966</td>
<td>0.00418</td>
<td>0.00392</td>
<td>0.00022</td>
<td>0.00525</td>
</tr>
</tbody>
</table>

\[ E(\sigma^4) = (0.00884 + (0.97373)^2)\sigma^2/(0.98932)^2 = 0.95995. \]

### Calculating Scheme for the Ratio of Expected Squared Lengths

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i(n_i) )</th>
<th>( \delta_{i1} )</th>
<th>( \delta_{21} )</th>
<th>( u_i )</th>
<th>( \delta_i )</th>
<th>( \delta_{i1}^2 )</th>
<th>( \delta_{21}^2 )</th>
<th>( u_i^2 )</th>
<th>( u_i^2 - u_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>1</td>
<td>26</td>
<td>50</td>
<td>0.27108</td>
<td>0.00228</td>
<td>0.2719</td>
<td>0.87020</td>
<td>0.3356</td>
<td>0.00215</td>
<td>0.34227</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>60</td>
<td>0.10580</td>
<td>0.00438</td>
<td>0.5707</td>
<td>0.65376</td>
<td>0.19992</td>
<td>0.00435</td>
<td>0.21998</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>80</td>
<td>0.23351</td>
<td>0.00609</td>
<td>0.9030</td>
<td>0.72642</td>
<td>0.15404</td>
<td>0.00601</td>
<td>0.15408</td>
</tr>
<tr>
<td>4</td>
<td>78</td>
<td>90</td>
<td>0.37774</td>
<td>0.01021</td>
<td>1.2771</td>
<td>0.32245</td>
<td>0.12565</td>
<td>0.01015</td>
<td>0.13670</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>50</td>
<td>0.41725</td>
<td>0.01849</td>
<td>1.7043</td>
<td>0.22349</td>
<td>0.09596</td>
<td>0.01839</td>
<td>0.11426</td>
</tr>
<tr>
<td>6</td>
<td>97</td>
<td>45</td>
<td>0.44453</td>
<td>0.02869</td>
<td>2.2030</td>
<td>0.14022</td>
<td>0.06133</td>
<td>0.02856</td>
<td>0.06634</td>
</tr>
<tr>
<td>7</td>
<td>103</td>
<td>74</td>
<td>0.65321</td>
<td>0.07399</td>
<td>2.8042</td>
<td>0.08054</td>
<td>0.05261</td>
<td>0.00043</td>
<td>0.06712</td>
</tr>
<tr>
<td>8</td>
<td>132</td>
<td>132</td>
<td>0.61667</td>
<td>0.13028</td>
<td>3.5581</td>
<td>0.04057</td>
<td>0.02590</td>
<td>0.00022</td>
<td>0.02747</td>
</tr>
<tr>
<td>9</td>
<td>108</td>
<td>161</td>
<td>0.83333</td>
<td>0.36111</td>
<td>4.5700</td>
<td>0.03614</td>
<td>0.01368</td>
<td>0.00010</td>
<td>0.01702</td>
</tr>
<tr>
<td>10</td>
<td>109</td>
<td>160</td>
<td>1.00000</td>
<td>1.00000</td>
<td>6.1966</td>
<td>0.00418</td>
<td>0.00392</td>
<td>0.00022</td>
<td>0.00525</td>
</tr>
</tbody>
</table>

\[ E(\sigma^6) = (0.00884 + (0.97373)^2)\sigma^2/(0.98932)^2 = 0.95995. \]
Figure 1—Approximate Power Functions for Mine Disaster Data: $k = 10$, $n = 109$, Optimum 10 quantiles.

References


