Economic Design of Attribute Control Charts

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This paper proposes a production model for the current control of the number of defectives by an np-chart. An algorithm for determining the most economic control parameters is presented. An exceedingly simple procedure is suggested for designing control plans which approximate in cost the most economic control plans. This procedure is applicable at the workshop level.

Key Words
Attributes
Control Charts
Process Control
NP-Charts
Proportion of Defectives
Economic Design

1. INTRODUCTION

Duncan (1956) and Cowden (1957; Chapter 29) independently pioneered the study of the design of X-charts by an economic approach. Duncan's model has received particular attention and a considerable amount of work has been developed from it since, e.g. Goel, et al. (1968), Duncan (1971), Gibra (1971), Chiu (1973), and Chiu and Wetherill (1974).

Relatively recently, Taylor (1968) has started some work on the economic design of cusum charts and Goel (1968) has made a comprehensive comparison between X-charts and cusum charts. Alongside, in the fields of economic acceptance sampling and economic continuous sampling many fruitful results have also been obtained, e.g. Hald (1968) and Hillier (1964).

It is to be regretted that the p-charts and np-charts for attribute surveillance have been largely neglected, despite the fact that in practice, they are in even wider application than X-charts because of the relative simplicity in handling an attribute quality characteristic than in handling a variable quality characteristic.

This paper proposes a production model for the current control of the number of defectives by an np-chart. We shall not discuss p-charts as they are equivalent to np-charts. A loss-cost function is established and the method of minimizing it is studied, which leads to an algorithm for the determination of the most economic control parameters.

Finally a very simple scheme is proposed for designing a control plan whose loss-cost is close to the minimum. This procedure is suitable for application at the workshop level.

2. FORMULATION OF THE PROBLEM

2.1 Various process models in the literature

In connection with the economic design of control charts, five production models have been studied. Cowden's (1957) model assumes that every morning the production starts in an unknown state. If a trouble is detected and corrected later in the day due to the operation of an X-chart, the process is free from further troubles. Because of this last assumption, Cowden's model is not suitable for the study of control charts, as the manufacturer may simply examine his process every morning, correct any trouble if found, and then start the production of the day without using any control chart.

Knappenberger and Grandage's (1969) model involves too many assumptions in formulating the cost function, especially the use of asymptotic, time-independent probabilities to replace the true process-state probabilities. The application of this model is likely to be restricted. Baker (1971) studies a model in which the time of 'in-control' depends on the number of false alarms. This situation is not general enough to warrant much consideration. We shall be more interested in the following two models. Duncan's (1956) model is simple and practical, but not sufficiently general, as it does not allow the process to be shut down when a search for the assignable cause is being carried out, and it does not include the time and cost of repairing the process if it is found to be out of control. Taylor's (1968) model considers the above elements which Duncan has left out, but omits the cost of sampling and assumes the effect of the assignable cause to be a function of the sample size. These two assumptions
are hardly very practical, as one cannot determine the most economic sample size and, if one changes the sample size, the effect of the assignable cause is also changed. In this paper, we propose an alternate model, which is a modification of each of the last two models, for an economic investigation of np-charts.

2.2 The production process and the loss-cost structure

The process is assumed to start in a state of 'in control', having a proportion of defective items \( p_0 \), where \( p_0 \) is a known constant. There is a single assignable cause of variation which has the effect of increasing the proportion of defective items to \( p_1 \), where \( p_1 \) is a known constant, and is bigger than \( p_0 \). The assignable cause occurs at the rate \( \lambda \) per hour of operating time, and the operating time until its occurrence is assumed to have an exponential distribution. Samples of size \( n \) are drawn every \( h \) hours of production and the process is undisturbed if the number of defectives found in the sample is \( d \) or less. The number \( d \) may be called the acceptance number. If the number of defectives found in a sample exceeds \( d \), the process is stopped and a search for the assignable cause is undertaken. An expected time of \( t_1 \) hours and an average search cost of \( A_1 \) are required if the assignable cause does not exist. After the search, the process proceeds and the points in operating time for the taking of samples continue the same as if there had been no interruption. If the assignable cause actually exists, it takes an average time of \( t_2 \) hours and an average cost of \( A_2 \) to discover it and to restore the process to the in-control state. The production starts anew in control after the adjustment.

In the above description, it is assumed that the production speed is so high that practically the assignable cause does not occur during the taking of a sample. This assumption is sometimes debatable as the sample size in attribute sampling is often large, but we shall accept this for simplicity. Also, we may note that in the np-chart that we are operating, the upper control limit is just less than \( d + 1 \) and the lower control limit is always set equal to zero. Finally, in the present model the time taken to test the sample and plot the sample point on the control chart is assumed to be negligible.

Let \( V_o \) denote the profit per hour earned by the process operating in control and \( V_t \) denote the profit per hour earned by it operating out of control. Finally, the cost of taking a sample and inspecting it is assumed to be linear in \( n \), namely, \( b + cn \).

2.3 Formulation of the lost-cost function

Let \( \alpha \) be the probability that the number of defectives in a sample will exceed \( d \) when the process is in control, and let \( P \) be the corresponding probability when the process is out of control. Then we have

\[
\alpha = 1 - \sum_{x=0}^{d} \binom{n}{x} p_0^x (1 - p_0)^{n-x}
\]

and

\[
P = 1 - \sum_{x=0}^{d} \binom{n}{x} p_1^x (1 - p_1)^{n-x}.
\]

A production cycle is defined to be the time period from the beginning of the production (or after an adjustment) to the detection and elimination of an assignable cause. The assignable cause occurs somewhere, say, between the \( m \)-th and the \((m + 1)\)-st samples, in the cycle. Then the average time of occurrence within this interval is:

\[
\tau = \int_{m \cdot h}^{(m+1) \cdot h} \exp(-\lambda y) \lambda(y-mh) \, dy
\]

\[
\int_{m \cdot h}^{(m+1) \cdot h} \exp(-\lambda y) \lambda \, dy
\]

\[
= \{1 - (1 + \lambda h) \exp(-\lambda h)\}/(\lambda - \lambda \exp(-\lambda h))
\]

Following an argument similar to Duncan (1956, pp. 230–231) we see that the average length of a production cycle consists of four parts: (a) the in-control period, \( 1/\lambda \), (b) the out-of-control period, \( h/P - \tau \), (c) the search time due to false alarms, \( A_0(1/\lambda - \tau)/h \), and (d) the search and adjustment time due to the true alarm, \( t_1 \). This gives the average length of a production cycle to be:

\[
1/\lambda + h/P - \tau + A_0(1/\lambda - \tau)/h + t_1.
\]

Similarly, the expected net profit derived from a production cycle is:

\[
V_o/\lambda + V_t(h/P - \tau) - \alpha A_0(1/\lambda - \tau)/h - A_1
\]

\[-(b + cn)(1/\lambda + h/P - \tau)/h.
\]

Hence the average net profit per hour is simply:

\[
I = \text{Expression (5)}/\text{Expression (4)}.
\]

Defining \( F \) by \( I = V_o - F \) we have, after suitable simplification,

\[
F = \frac{\lambda MB_0 + TB_0 + \lambda W + (b + cn)(1 + \lambda B_1)/h}{1 + \lambda B_1 + t_0 B_0 + \lambda t_1},
\]

where

\[
B_0 = \alpha(1 - \lambda \tau)/h, \quad B_1 = h/P - \tau,
\]

* The factor \( \alpha(1/\lambda - \tau)/h \) here is algebraically equivalent to Duncan's \( \alpha \exp(-\lambda h)/(1 - \exp(-\lambda h)) \).
and

\[ M = V_0 - V_1, \quad T = A_0 + V_{d_0}, \]
\[ W = A_1 + V_{d_1}. \]  

(8)

The function \( F \) represents the loss cost per hour for the present model. It should be noted that \( F \) is a function of the three control scheme parameters, \( h, n, \) and \( d \), because \( \alpha \) and \( P \) are functions of \( n \) and \( d \). The problem of an economic design is the search for those values of \( h, n, \) and \( d \), for which \( F \) is a minimum. In the rest of this paper, we shall study two methods of determining these control parameters: a two-dimensional Fibonacci search and a simplified procedure.

### 3. An Algorithm for Minimizing the Loss-Cost Function

For any pair of specified values of \( d \) and \( n \), the (conditionally) most economic value of \( h \) can be easily determined in the following way. Equating to zero the partial derivative \( \partial F / \partial h \) we have

\[
\lambda h^2 (M + t_u B_o M + \lambda t_u M - B_o T - \lambda W) \frac{\partial B_o}{\partial h} \\
+ h^2 (T + \lambda B_o T + \lambda T - \lambda B_o M - \lambda_0 W) \frac{\partial B_o}{\partial h} \\
- (b + cn)(1 + \lambda B_1 + t_u B_0 + \lambda t_u)(1 + \lambda B_1) \\
+ t_u h(1 + \lambda B_1) \frac{\partial B_o}{\partial h} \\
- \lambda h(t_u B_0 + \lambda t_u) \frac{\partial B_i}{\partial h}) = 0, \quad (9)
\]

where

\[
\frac{\partial B_o}{\partial h} = \frac{\alpha^2}{2 - \exp(\alpha h) - \exp(-\alpha h)}, \quad (10)
\]
\[
\frac{\partial B_i}{\partial h} = \frac{1/P + 1/\left|\exp(\beta h) - 1\right|}{1/P + 1/\left|\exp(\beta h) - 1\right|} \\
+ \lambda h/\left|2 - \exp(\beta h) - \exp(-\beta h)\right|. \quad (11)
\]

In the next section we shall show (see equation 19) that the quantity

\[
\left[ (\alpha T + b + cn)/(\lambda M (1/P - 1/2)) \right]^{1/2}
\]

in general approximates the true root of \( h \) given by equation (9). Thus equation (9) can be easily solved numerically on a computer by using this quantity as an initial root. A subroutine based on Newton’s iteration method is adequate because the left hand side of (9) is a well behaved function of \( h \) and its derivative can be expressed in an explicit mathematical form.

No analytic method exists for the determination of the most economic values of \( n \) and \( d \), for the equations \( \partial F / \partial n = 0 \) and \( \partial F / \partial d = 0 \) are intractable and the latter in fact does not have a convenient mathematical expression. A direct search method is thus preferred, especially from the point of view that both \( n \) and \( d \) are integers. While many search methods are available, a two-dimensional Fibonacci search appears to be specially efficient, in the sense that only a small number of evaluations of the loss-cost function \( F \) is required. This method has been discussed by Sugie (1964) and Dixon (1972). The procedure is explained below.

We first choose a search area of reasonable size, e.g. \( 0 \leq d \leq 12 \) and \( 1 \leq n \leq 144 \), in the \((d, n)\)-plane, around a guessed minimum position. This position may be determined by the simplified scheme proposed in the next section. The choice of the search area will be illustrated in Example 1. Then a Fibonacci search for the value of \( n \), given \( d \), is nested in a Fibonacci search for the value of \( d \). In each inner loop, \( d \) is given and the conditionally most economic value of \( n \) is determined according to the flow chart by Dixon (1972, p. 23), as follows. In each step, an \( n \) value is nominated according to a fixed rule prescribed by the Fibonacci numbers. The probabilities \( \alpha \) and \( P \) of equations (1) and (2), respectively, are calculated; then equation (9) is solved for \( h \) with the help of the initial root mentioned above; and the value of \( F \) can finally be calculated, for a comparison with the result of the previous step. The current values of \( n, h, \) and \( F \) replace those already stored in the computer if and only if the current \( F \) value is smaller.

In each step of the outer loop a \( d \) value is nominated according to the prescribed rule. The conditionally most economic values of \( n, h, \) and \( F \) are then determined as above. The current values of \( d, n, h, \) and \( F \) replace those already stored in the computer if and only if the current \( F \) is smaller. The final step gives a pair of values \( d_o \) and \( n_o \) whose status as a minimum position is further checked by verifying that the value of \( F \) for \((d_o, n_o)\) is less than the local minima in the rows \( d_o \pm 1 \) and the columns \( n_o \pm 1 \).

A program comprising eight subroutines according to the algorithm just outlined has been written. This program is quite efficient as it usually requires only 50–100 evaluations of \( F \) to locate the minimum position \((d_o, n_o, h_o)\), \( h_o \) being accurate to five decimal places. Each determination of a most economic plan takes approximately 0.3–0.5 minute on an ICL 4–50 computer, using double precision arithmetic. In the following example some computer output is presented to show how the algorithm works.

**Example 1.** Suppose the values of the cost-risk parameters for a process are as given below: \( p_0 = 0.015 \), \( p_1 = 0.10 \), \( \lambda = 0.01 \), \( V_0 = 150 \), \( V_1 = 50 \), \( A_0 = 10 \), \( t_0 = 0.1 \), \( A_1 = 30 \), \( t_1 = 0.3 \), \( b = 0.5 \), and \( c = 0.01 \). We are interested in finding the values of \( d, n, \) and \( h \) which minimize the loss-cost \( F \).

A result of the next section may be anticipated that the simplified scheme gives two nearly most economic control parameter values: \( d = 4 \) and \( n = ... \)
The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots, so that we may choose the search area: \(1 \leq d + 1 \leq 13, 1 \leq n \leq 144\), which contains the initial position \(d + 1 = 5, n = 80\), and whose boundaries take values of some Fibonacci numbers. (We have used \(d + 1\) instead of \(d\) because the smallest Fibonacci number is 1 but the smallest acceptance number is 0.) The results of the two-dimensional Fibonacci search are summarized in Table 1.

In Table 1 each column represents an inner loop and the results of the columns represent the outer loop. We see that the most economic values of the control parameters are: \(d = 4, n = 77\), together with \(h = 1.53\) (not shown in the table). The minimum loss-cost value is \(F = 2.60156\).

4. A SIMPLIFIED ECONOMIC SCHEME

The above algorithm gives the exact, most economic parameter values as no approximation has been introduced in any stage. The method is academically interesting but may be practically difficult as it does not appear to be concise to a practitioner with limited mathematical and programming knowledge. We therefore propose a simplified scheme in this section for practical application as well as for the provision of an initial position for the two-dimensional Fibonacci search.

The simplified scheme is based on a specification of the probability of a true alarm, \(P\). Duncan (1965; p. 426) has observed that in the economic design of \(X\)-charts the average run length in the out-of-control state often has a value between 1 and 2. Chiu and Wetherill (1974) propose a simplified economic scheme for the design of \(X\)-charts by putting \(P = 0.9\), and have found that their scheme often leads to plans which approximate the most economic plans in cost. In a similar spirit, we shall specify

\[
P = 0.90
\]

for the model and the np-chart control procedure of Section 2.2. This condition defines a relationship between \(n\) and \(d\) because of (2). As far as the author's extensive numerical investigations have indicated this relationship is often nearly most economic. However, we shall view the procedure which will be developed from (12) as a semi-economic scheme. A specification of \(P\) means a selection of the level of protection against the deteriorated quality: \(P = 0.9\) will enable the manufacturer to detect an assignable cause rather quickly, on the average about 1.1 samples after its occurrence, so as to reduce the loss due to the prolonged production of a large proportion of defectives. It should be emphasized that, although our scheme possesses it, the property that the resulting loss-cost is often close to the exact minimum loss-cost is desirable, but not essential to a semi-economic scheme.

We shall introduce several approximations to the expression of \(F\) in (6) in order to arrive at a simple scheme. Since in most practical situations \(p_0\) and \(p_1\) are small and \(n\) is large, we may use the Poisson distribution to approximate the binomial distribution in (1) and (2). Thus:

\[
\alpha = 1 - \exp \left( -m_0 \sum_{x=0}^{d} \frac{m_0^x}{x!} \right)
\]

\[
= 1 - \exp \left( -m_0(p_0/p_1) \sum_{x=0}^{d} \frac{m_1^x(p_0/p_1)^x}{x!} \right)
\]

and

\[
P = 1 - \exp \left( -m_1 \sum_{x=0}^{d} \frac{m_1^x}{x!} \right),
\]

where \(m_0 = np_0\), and \(m_1 = np_1\).

The solutions of equation (12), based on the

![Table 1 — Results of the Fibonacci Search in the (d, n)-plane for Example 1](image_url)

Notes: (a) The underlined figures are the minimum values in the columns. (b) The starred figure is the overall minimum value.

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expression for \( F \) in (14), for various values of \( d \) are given in the first two columns of Table 2.

Regarding expression (3) Duncan (1956, p. 229) has shown that \( \tau \) is very well approximated by the expressions (13), (14), and (16) are used.

In the expression for \( F \) in (6), note that in practice, \( \lambda \) and \( \alpha_0 \) are small, they are typically of order, say, 0.01. Hence in the denominator of \( F \) in (6), the quantity \( \lambda \beta_1 + \lambda \beta_0 + \lambda \lambda \) is small compared with unity and can be omitted; also in the numerator, the term \( \lambda \beta_1 \) is small compared with unity and can be omitted from the last pair of brackets. That these omissions take place in the numerator and the denominator partially balances the small inaccuracy incurred.

Thus we have:

\[
F = F' = \lambda MB_1 + TB_0 + \lambda W + (b + cn)/h. \tag{17}
\]

We understand that in (17), the approximate expressions (13), (14), and (16) are used.

To find another relationship between \( n \) and \( d \) in conjunction with condition (12), we proceed as follows. For every value of \( d \), there is a value of \( m_1 \), which is given by (12) and has been listed in the second column of Table 2. We may define the following incremental quantities:

\[
\Delta m_1 = m_1(d + 1) - m_1(d),
\]

\[
\Delta F' = (1/h - \lambda/2 + \lambda h/12)T \Delta m_1.
\]

To minimize \( F' \) with respect to \( d \) subject to (12) we have:

\[
\Delta_{d-1} F' < 0 \leq \Delta_{d} F',
\]

which is equivalent to:

\[
(1/h - \lambda/2 + \lambda h/12) \Delta_{d-1} m_1 < -c/(h P_0) \leq (1/h - \lambda/2 + \lambda h/12) \Delta_{d} m_1.
\]

Omitting small terms of the order of \( c(A_1 A_1 m_1) \)

TABLE 2—Values of \(-A_1P_1/A_0P_0\) for \( P = 0.90\)

<table>
<thead>
<tr>
<th>d</th>
<th>( P_0/P_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>17.2 10.0 7.0 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.80</td>
</tr>
<tr>
<td>1.0</td>
<td>19.2 13.8 11.3 10.0 9.2 10.0 11.2 12.9 15.1 18.1 22.1 27.5 35.2 64.6</td>
</tr>
<tr>
<td>2</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>2.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>3.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>4.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>5.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>6.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>7.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
<tr>
<td>8.0</td>
<td>24771 2757.1 109.2 10.2 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0 10.0</td>
</tr>
</tbody>
</table>

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and higher, and re-arranging each inequality, we have:

\[-(\Delta_{n-1}m_i/\Delta_{n-1}a) < Tp_i/c \leq -(\Delta m_i/\Delta a)\]  

(18)

The solution of (18) can be obtained with the help of Table 2, which gives the values of \(-(\Delta m_i/\Delta a)\) in terms of \(d\). The construction of Table 2 is quite straightforward on a computer.

Finally for any given values of \(n\) and \(d\) the value of \(h\) that minimizes \(F\) can be obtained by solving the equation \(\partial F'/\partial h = 0\) which approximates equation (9). Solving it, and omitting terms in \(\lambda^2\) and \(\lambda a^2\), we have:

\[n = [(\alpha T + b + cn)/\lambda M(1/P - 1/2)]^{1/2}\]  

(19)

which, in general, satisfactorily approximates the root of (9). We may note that unless we put \(P = 0.90\), (19) is independent of condition (12) from the way it has been derived, and hence is suitable for being an initial root for solving (9) in the algorithm of the last section.

In conclusion we have arrived at a very simple semi-economic scheme for determining values of \(d\), \(n (= m_i/p_i)\), and \(h\), based on equations (12), (19), and inequality (18). The scheme which uses Table 2 is applicable at the workshop level. An illustrative example is given below.

Example 2. Consider the data of Example 1: \(p = 0.015\), \(p_1 = 0.10\), \(\lambda = 0.01\), \(V_0 = 150\), \(V_1 = 50\), \(A_0 = 10\), \(t_o = 0.1\), \(A_1 = 30\), \(t_i = 0.3\), \(b = 0.5\), and \(c = 0.01\). The definitions of (8) yield: \(M = 100\), \(T = 25\), and \(W = 75\). To find the control plan by the simplified scheme developed above, we proceed as follows.

First step (Calculate \(p_0/p_i\) and \(T p_i/c\)): From the data, we have: \(p_0/p_i = 0.15\), and \(T p_i/c = 250\).

Second step (Determine \(d\) and \(n\)): From Table 2, in the column for \(p_0/p_i = 0.15\) we find that \(d = 4\) is the solution to (18) because \(115.2 < 250 \leq 277.5\). Correspondingly, \(m_i = 7.994\), which yields \(n = m_i/p_i = 80\).

Third step (Evaluate \(h\)): We know that \(P = 0.90\).

From \(m_0 = n p_0 = 1.2\) and expression (13), we have \(\alpha = 0.00775\). Substituting these quantities into (19), we obtain \(h = 1.56\).

Fourth step (Estimate \(F\)-optional): Substituting the control parameter values \(d = 4\), \(n = 80\), and \(h = 1.56\) into (6) and using the exact expressions (1), (2), and (7), we obtain \(F = 2.60485\).

A comparison between this \(F\) value and the exact minimum value of \(F\) (= 2.60156) given at the end of Example 1 shows that the difference is negligible.

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Appendix

List of Symbols

\[p_o = \text{proportion of defectives for the in-control state.}\]

\[p_1 = \text{proportion of defectives for the out-of-control state.}\]

\[\lambda = \text{average rate of occurrence per hour of the assignable cause.}\]

\[t_o = \text{expected duration for a search after a false alarm.}\]

\[t_i = \text{expected duration for a search and adjustment after a true alarm.}\]

\[A_o = \text{expected cost for a search after a false alarm.}\]

\[A_i = \text{expected cost for a search and adjustment after a true alarm.}\]

\[V_o = \text{profit per hour earned by an in-control process.}\]

\[V_1 = \text{profit per hour earned by an out-of-control process.}\]

\[b = \text{overhead cost of taking and testing a sample.}\]

\[c = \text{variable cost per item of sampling and testing.}\]

\[\alpha = \text{probability on a single sample of UCL being exceeded in the absence of an assignable cause.}\]

\[P = \text{probability on a single sample of UCL being exceeded in the presence of an assignable cause.}\]

\[\tau = \text{average time between the sample taken just before the occurrence of an assignable cause and the occurrence itself.}\]

\[\beta = (1 - \lambda\tau)/h.\]

\[B_i = h/P - \tau.\]

\[I = \text{average net profit per hour when a given control procedure is operated.}\]

\[F = \text{average loss-cost per hour of operating the control procedure.}\]

\[F' = \text{an approximate expression of } F \text{ used to derive the simplified scheme.}\]

References


